HYPERS301.x Hypersonics - From Shock Waves to Scramjets

Summary of Some Useful Equations

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Section 1 – Introduction

Mach number:
The Mach number, M , of an object moving at speed, v , through a gas that has a sound speed, a , is defined as

$$M = \frac{v}{a}$$

Flow Regimes in Terms of Mach Number:
Subsonic: $0 < M < 1$
Transonic: $0.8 < M < 1.2$
Supersonic: $M > 1$
Hypersonic: $M > 5$

Mass Ejection:

When an object is ejecting some of its mass at a relative velocity of $\vec{v}_p$ and at a rate of $\frac{dm_p}{dt}$ the effective force on the object, $\vec{F}$, is given by

$$\vec{F} = -\frac{dm_p}{dt}\vec{v}_p$$

Note here that the force is in the opposite direction to that of the ejected mass.

The Rocket Equation:
Two simple forms of the rocket equation are

$$\Delta v = v_p \ln \frac{m_0}{m_f}$$

$$\frac{m_f}{m_0} = e^{-\frac{\Delta v}{v_p}}$$

where

m_0 is the initial mass of the rocket,

m_f is the final mass of the rocket,

v_p is the speed with which the propellant is ejected relative to the rocket, and

Δv is the overall change in the speed of the rocket.

In order to account for the effects of gravity and aerodynamic drag, Δv can be replaced by an "effective" change in velocity of Δv_{eff} .

Specific Impulse:
The Specific Impulse, I_{sp}, of a propulsion system is defined as the thrust per unit weight flow rate of the fuel. Therefore, if the thrust is T, if the mass flow rate of fuel is $\frac{dm_p}{dt}$ and the acceleration due to gravity is g, then

$$I_{sp} = \frac{T}{g\frac{dm_p}{dt}}$$

If we write $\frac{dm_p}{dt}$ as $\dot{m}_p$, then we can write the specific impulse also as

$$I_{sp} = \frac{T}{g\dot{m}_p}$$

For a rocket ejecting propellant continuously at a relative speed of v_p, the specific impulse can be written as

$$I_{sp} = \frac{v_p}{g}$$

Single Stage of a Rockets:
For any stage of a rocket

$$m_0 = m_{PL} + m_p + m_s$$

and

$$m_f = m_{PL} + m_s$$

where

m_0 is the total initial mass of that stage of the rocket,

m_{PL} is the mass of the payload for that stage,

m_p is the mass of the propellant for that stage, and

m_s is the mass of everything else for that stage (structure, engines, etc.). We refer to this as the "structural mass"

Two Stage Rockets:
For a two-stage rocket

$$m_{01} = m_{PL} + m_{s2} + m_{p2} + m_{p1} + m_{s1}$$
$$m_{f1} = m_{PL} + m_{s2} + m_{p2} + m_{s1}$$
$$m_{02} = m_{PL} + m_{s2} + m_{p2}$$

and

$$m_{f2} = m_{PL} + m_{s2}$$

where

m_{01} is the total initial mass of stage 1,

m_{PL} is the mass of the intended payload,

m_{p1} is the mass of the propellant for stage 1,

m_{s1} is the structural mass for stage 1,

m_{02} is the total initial mass of stage 2,

m_{p2} is the mass of the propellant for stage 2, and

m_{s2} is the structural mass for stage 2.

Note that it is assumed that the structural mass for stage 1 is discarded before the rocket motor for stage 2 is fired.

Section 2 – Isentropic Flow

Mach number: $M = V/a$

Speed of sound: $a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$, $a = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}}$

Ideal gas law: $p = \rho RT$

Specific heats (perfect gas): $dh = c_p dT$, $du = c_v dT$, $\gamma = \frac{c_p}{c_v}$, $R = c_p - c_v$

Isentropic flow relations: $\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$, $\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$

Stagnation enthalpy: $h_0 = h + \frac{1}{2}V^2$

Stagnation temperature: $T_0 = T + \frac{1}{2c_p}V^2$, $\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$

Stagnation pressure: $\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$

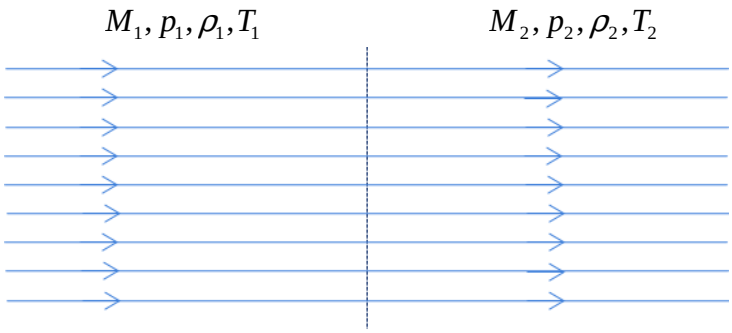
Ratio of current area to critical area: $\frac{A}{A^*} = \frac{1}{M}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}$

Compressible flow calculator:
Compressible Aerodynamics Calculator
http://www.dept.aoe.vt.edu/~devenpor/aoe3114/calc.html
Javascript by William J. Devenport, Department of Aerospace and Ocean Engineering, Virginia Tech. Modified by Adam Ford to include Fanno Flow and Rayleigh Flow 7th February 2008.

Section 3 – Shock Waves

Summary of important results from this unit:

1. Normal Shock Waves



Ratio of static density across a normal shock wave:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$$

Ratio of static pressure across a normal shock wave:

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma+1}$$

Ratio of static temperature across a normal shock wave:

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1^2 - \gamma + 1\right]\left[(\gamma-1)M_1^2 + 2\right]}{(\gamma+1)^2 M_1^2}$$

Mach number downstream of a normal shock wave:

$$M_2 = \left\{ \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}} \right\}^{\frac{1}{2}}$$

Ratio of stagnation pressure across a normal shock wave:

$$\frac{p_{02}}{p_{01}} = \left\{ \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \right\}^{\frac{\gamma}{\gamma-1}} \left\{ \frac{\gamma+1}{2\gamma M_1^2 - \gamma + 1} \right\}^{\frac{1}{\gamma-1}}$$

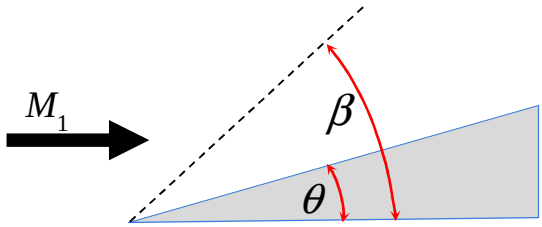
Change in entropy across a normal shock wave:

$$\Delta s = -R \ln \frac{p_{02}}{p_{01}}$$

Stagnation temperature remains constant across a normal shock wave:

$$T_{01} = T_{02}$$

2. Oblique Shock Waves



$$M_{n1} = M_1 \sin\beta$$

$$M_{n2} = M_2 \sin(\beta - \theta)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2} = \frac{V_{n1}}{V_{n2}} = \frac{\tan\beta}{\tan(\beta-\theta)}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_{n1}^2 - \gamma + 1}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_{n1}^2 - \gamma + 1\right]\left[(\gamma-1)M_{n1}^2 + 2\right]}{(\gamma+1)^2 M_{n1}^2}$$

$$M_{n2} = \left\{\frac{1+\frac{\gamma-1}{2}M_{n1}^2}{\gamma M_{n1}^2 - \frac{\gamma-1}{2}}\right\}^{\frac{1}{2}}$$

$$\frac{\tan\beta}{\tan(\beta-\theta)} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2}$$

3. Useful Tables and calculator

Tabulated data for compressible flows:
NACA Report 1135 – Equations, Tables and Charts for Compressible Flow
http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19930091059_1993091059.pdf

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Section 4 – Combustors in Scramjets

Fanno Flow Parameter Relations

Choked Flow at Location L^*: $\frac{4fL^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{(\gamma+1)}{2\gamma} ln\left(\frac{M^2}{\frac{2}{(\gamma+1)}\left(1+\frac{\gamma-1}{2}M^2\right)}\right)$

Temperature Ratio: $\frac{T}{T^*} = \left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)$

Pressure Ratio: $\frac{p}{p^*} = \frac{1}{M}\sqrt{\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)}$

Density Ratio: $\frac{\rho}{\rho^*} = \frac{1}{M}\sqrt{\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)}$

Total Pressure Ratio: $\frac{p_0}{p_0^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$

Rayleigh Flow Relations

Total Temperature Ratio: $\frac{T_0}{T_0^*} = 2(1+\gamma)\left[\frac{\left(1+\frac{\gamma-1}{2}M^2\right)M^2}{(1+\gamma M^2)^2}\right]$

Temperature Ratio: $\frac{T}{T^*} = \left(\frac{(\gamma+1)M}{(1+\gamma M^2)}\right)^2$

Pressure Ratio: $\frac{p}{p^*} = \frac{(\gamma+1)}{(1+\gamma M^2)}$

Density Ratio: $\frac{\rho}{\rho^*} = \frac{(1+\gamma M^2)}{(\gamma+1)M^2}$

Total Pressure Ratio: $\frac{p_0}{p_0^*} = \left(\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma}{\gamma-1}}\frac{(\gamma+1)}{(1+\gamma M^2)}$

Stoichiometry Relations

Number of moles of $O_2(a)$ to fully react an arbitrary Hydrocarbon fuel (C_xH_y): $a = x + \frac{y}{4}$

Molar Air-Fuel Stoichiometric Ratio: $n)_{stoic.} = 4.76 \times a$

Mass based Air-Fuel Stoichiometric Ratio : $m)_{stoic.} = n)_{stoic.} \times \left(\frac{MW_{Air}}{MW_{Fuel}}\right)$

Equivalence Ratio: $\phi = \frac{(A/F)_{stoic.}}{A/F} = \frac{n)_{stoic.}}{n} = \frac{n)_{stoic.} \times (MW_{Air}/MW_{Fuel})}{n \times (MW_{Air}/MW_{Fuel})} = \frac{m)_{stoic.}}{m}$

Mass based Air-Fuel Ratio: $m = \frac{m)_{stoic.}}{\phi}$

Mass of fuel: $m_{Fuel} = \frac{m_{Air}}{m}$

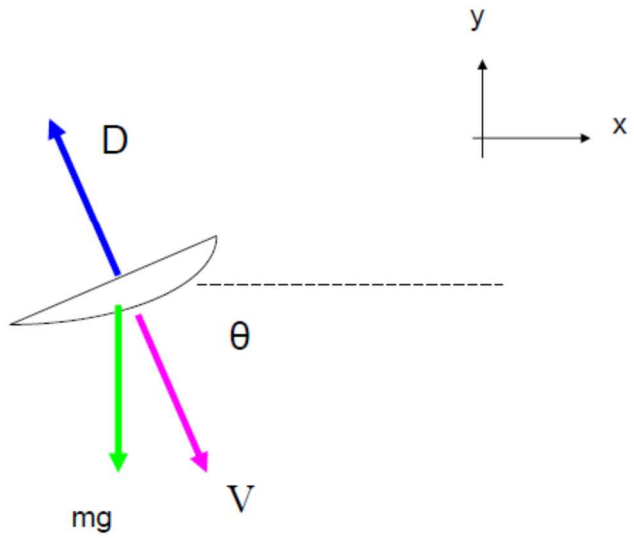
Total Potential Heat Released: $= \frac{m_{air}}{m} \times LHV$

Actual Heat Released: $= \eta_{comb.} \times \frac{m_{air}}{m} \times LHV$

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Section 5 – Hypersonics

'Flat earth' trajectory



$$\frac{dx}{dt} = V\cos\theta$$

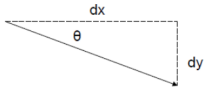
$$\frac{dy}{dt} = -V\sin\theta$$

$$m\frac{dV}{dt} = -D + mg\sin\theta$$

$$mV\frac{d\theta}{dt} = mg\cos\theta$$

$$\left(due\ to\ rotation\ of\ velocity\ vector\right)$$

- **Redundant variables exist, as x, y and t are not independent**



$$\frac{dx}{dy} = -\cot\theta$$

$$m\frac{dV}{dt} = m\frac{dV}{dy}\frac{dy}{dt} = -m\frac{dV}{dy}V\sin\theta$$

$$= -D + mg\sin\theta$$

$$\therefore mV\frac{dV}{dy} = \frac{D}{\sin\theta} - mg$$

$$mV\frac{d\theta}{dt} = mV\frac{d\theta}{dy}\frac{dy}{dt} = -mV^2\frac{d\theta}{dy}\sin\theta$$
$$= mg\cos\theta$$

$$\therefore \frac{d\theta}{dy} = -\frac{g}{V^2}\cot\theta$$

- **For most trajectories, V^2 is large, θ is small**
- **Therefore $\frac{d\theta}{dy}$ will be small**

- **Therefore a reasonable approximation for the early part of the trajectory will be given by assuming θ is constant**

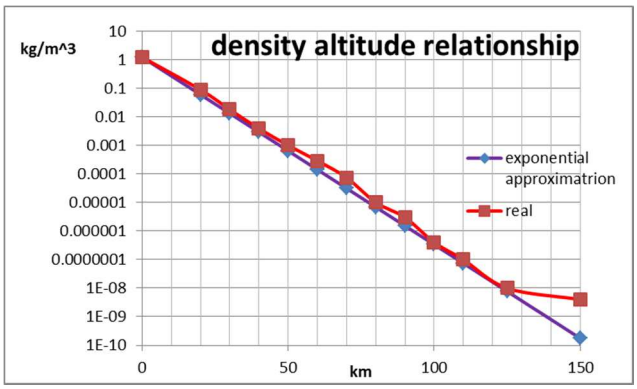
- **This is equivalent to assuming Drag >> mg**

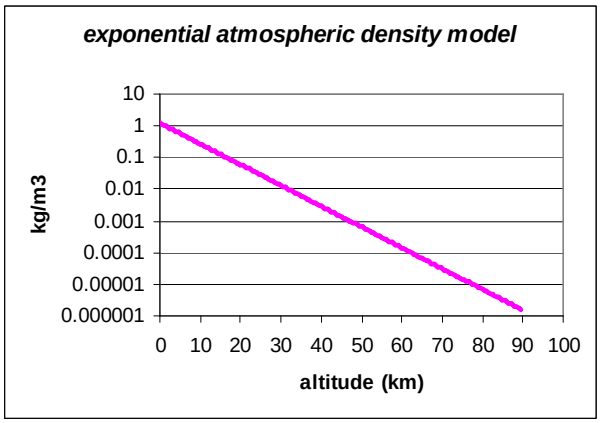
$$\therefore -\frac{dV}{dt} = -\frac{dy}{dt}\frac{dV}{dy} = V\sin\theta\frac{dV}{dy} = \frac{D}{m}$$
$$= C_D\frac{A\rho V^2}{2m}$$
$$\therefore \frac{1}{V}\frac{dV}{dy} = \frac{C_D A\rho}{2m\sin\theta}$$

This is directly integrable, if we know ρ(y)

Approximate atmospheric model

$$\rho = 1.22e^{\left(-\frac{y}{y_o}\right)}, (kg\,m^{-3})$$
with y_o being the 'scale height'
of 6.62 km for Earth





In the following equations the symbol ρ_0 represents the atmospheric density at ground level for which a nominal value of 1.22 kg m^{-3} is taken for this class.

$$\therefore \frac{1}{V}\frac{dV}{dy} = \frac{C_D A \rho_o e^{-\frac{y}{y_o}}}{2m\sin\theta}$$

$$\therefore \ln(V) = -\frac{C_D A \rho_o y_o e^{-\frac{y}{y_o}}}{2m\sin\theta} + const$$

$$\therefore V = const * \exp\left\{-\frac{C_D \rho_o A y_o}{2m\sin\theta} e^{-\frac{y}{y_o}}\right\}$$

Note that the vehicle never fully stops in this model, and that at the lower levels of velocity, the assumption that drag >> mg breaks down

- The integration constant may be found from the boundary condition that as y approaches ∞, ρ → 0 and V = V_E, the entry velocity, giving:

$$\therefore V = V_E \exp\left\{-\frac{C_D \rho_o A y_o}{2m\sin\theta} e^{-\frac{y}{y_o}}\right\}$$

Deceleration may be found by differentiating the expression for velocity, giving:

$$f = \frac{1}{g}\frac{dV}{dt} = \frac{1}{g}\frac{dV}{dy}\frac{dy}{dt} = \frac{1}{g}V\sin\theta\frac{dV}{dy}$$ (with f measured in 'g's')

$$\therefore f = \frac{C_D \rho_o A V_E^2}{2mg} e^{-\frac{y}{y_o}} \exp\left\{-\frac{C_D \rho_o A y_o}{m \sin\theta} e^{-\frac{y}{y_o}}\right\}$$

- ***This expression may be differentiated again to find where gmax occurs:***

$$y = y_o \ln\left\{\frac{C_D \rho_o A y_o}{m \sin\theta}\right\}$$

$$at\ this\ point,\ V = V_E e^{-0.5} = 0.61 V_E$$

$$and\ f_{\max} = -\left\{\frac{1}{g}\frac{dV}{dt}\right\}_{\max} = \frac{V_E^2 \sin\theta}{2 g e y_o}$$

- note that the maximum deceleration and the velocity at which it occurs are independent of vehicle characteristics
- the altitude is not
- the reentry angle is the primary variable for controlling forces

Noting that the generic dependency on heat transfer scales with: $\dot{q} = \sqrt{\rho}\, U^3$

We get:

$$\dot{q} \sim \sqrt{\rho_0} V_E^3 e^{-\frac{\beta y}{2}} \exp\left\{\frac{-3 C_D \rho_0 A}{2\beta m \sin\theta_E} e^{-\beta y}\right\}$$

This has a maximum value when:

$$y = \frac{1}{\beta} \ln\left\{\frac{3 C_D \rho_0 A}{\beta m \sin\theta_E}\right\}$$

$$and\ V = V_E e^{-\frac{1}{6}} = 0.85\, V_E$$

$$\tau = \mu \frac{dU}{dy} \sim \frac{\mu U}{\delta} = \frac{1}{2} \rho U^2 C_{fl}$$

$$\overset{*}{q} = k \frac{dT}{dy} \sim \frac{k T_r}{\delta}, \quad but \quad T_r \sim \frac{U^2}{2 C_p}$$

$$\therefore \quad \overset{*}{q} \sim \frac{k U^2}{2 C_p \delta} = \frac{k U}{2 C_p \mu} \frac{\mu U}{\delta}$$

$$= \frac{k}{C_p \mu} \frac{U}{2} \tau \approx \frac{1}{4} \rho U^3 C_{fl}$$

If Q is the total heat transferred to a vehicle during re-entry then:

$$\frac{dQ}{dt} = \oint_S \dot{q} \, dS = \frac{1}{4} \rho U^3 \oint_S C_{fl} \, dS$$

$$dt = -\frac{dy}{U \sin\theta_E}$$

$$\therefore \frac{dQ}{dy} = -\frac{dQ}{dt} \frac{1}{U \sin\theta_E} = -\frac{C_{f'} \rho U^2 S}{4 \sin\theta_E}$$

$$with \; C_{f'} = \frac{1}{S} \oint_S C_{fl} \frac{\rho}{\rho_l} \frac{U_l}{U} \, ds$$

$$\frac{dQ}{dy} = -\frac{C_{f'} \rho_0 U_E^2 S}{4 \sin\theta_E} e^{-\beta y} \exp\left\{ \frac{-C_D \rho_0 A}{\beta m \sin\theta_E} e^{-\beta y} \right\}$$

$$\therefore Q = \int_0^\infty \frac{dQ}{dy} \, dy = \frac{1}{4} \left(\frac{C_{f'} S}{C_D A} \right) m U_E^2 \left\{ 1 - \exp\left\{ \frac{-C_D \rho_0 A}{\beta m \sin\theta_E} \right\} \right\}$$

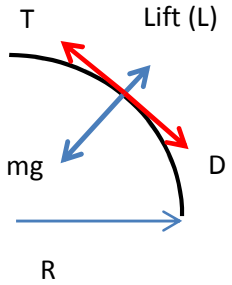
$$or \quad Q = \frac{m}{4} \left(\frac{C_{f'} S}{C_D A} \right) \left\{ U_E^2 - U_0^2 \right\} = total \;\; heat \;\; absorbed \;\; during \;\; entry$$

$$Or \quad Q = \frac{E}{2}\left(\frac{C_f \cdot S}{C_D A}\right)$$

$$where \;\; E \;=\; total \;\; kinetic \;\; energy \;\; lost \;\; by \;\; vehicle$$

The Cruise Corridor

When a vehicle travels at cruise, it flies at constant speed and altitude, and the sum of the forces on it are equal to the centripetal acceleration required to keep it moving in the arc of a circle. So thrust (T) must be equal to the drag (D), and the net force of gravity minus lift must match the required acceleration to maintain circular motion.



$$\frac{MV^2}{R} = Mg - L$$

$$D = L$$

Alternatively, using the velocity required to maintain constant altitude (ie circular orbit) in space, where there is no lift, given by $V_L = \sqrt{g\,R}$ (where g is the local gravitational acceleration, not the universal gravitational constant), and noting that the extent of the Earth's atmosphere (approximately 100 km) is small compared to the radius of the Earth, it is apparent that the value of 'g' doesn't change much over the expected altitude range of hypersonic flight, the lift required at cruise within the atmosphere can be approximated by:

$$\therefore \; L = m\left\{\frac{V_L^2}{R} - \frac{V^2}{R}\right\}$$

Section 6 –Scramjets

$$T = F_{out} - F_{in} - F_{add}$$

$$T = \{p_{out} A_{out} + (\dot{m}_{air} + \dot{m}_{fuel}) V_{out}\} - \{p_{in} A_{in} + \dot{m}_{air} V_{in}\} - F_{add}$$

$$\frac{T_{04}}{T_{03}} = 1 + \frac{\varphi f_{st} h_f \eta_c}{C_p T_{03}}$$

$$F_{in} = F_0 = p_0 A_0 (1 + \gamma M_0^2)$$

$$F_{out} = F_{10} = P_{10} A_{10} (1 + \gamma M_{10}^2)$$