Chapter 10

Spin

10.1 Spin $\frac{1}{2}$ as a Qubit

In this chapter we will explore quantum spin, which exhibits behavior that is intrinsically quantum mechanical. For our purposes the most important particles are electrons and protons which are spin 1/2 particles. Their spin state is a qubit, which can be in one of two orthogonal states: spin up denoted by $|\uparrow\rangle$ or spin down $|\downarrow\rangle$. By the superposition principle, the spin state of an electron or proton is thus $|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$. The spin state is an excellent way to implement a qubit in real life.

The Bloch Sphere

Before we can further understand spin, it is useful to digress into a 3-dimensional representation of a qubit via the "Bloch Sphere." This representation allows us to picture the state of a qubit as a point on the unit sphere in 3-dimensional Euclidean space $\mathbb{R}^3$. This convenient mapping between possible single-qubit states and the unit sphere is best explained by picture:

Choosing θ and ϕ as the usual spherical coordinates, every point (θ, ϕ) on the unit sphere represents a possible state of the qubit:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

And any possible state of a qubit (up to an overall multiplicative phase factor) is represented by a vector on this unit sphere.

What does the action of a single qubit gate correspond to in this Bloch sphere representation? The answer is natural and elegant: a single qubit gate is just a rotation of the Bloch Sphere (see homework). Let's do an example. Consider the Hadamard gate $\mathbf{H}$ that has been discussed in the past. (Note that $\mathbf{H}$ is not equal to the Hamiltonian in this case!)

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

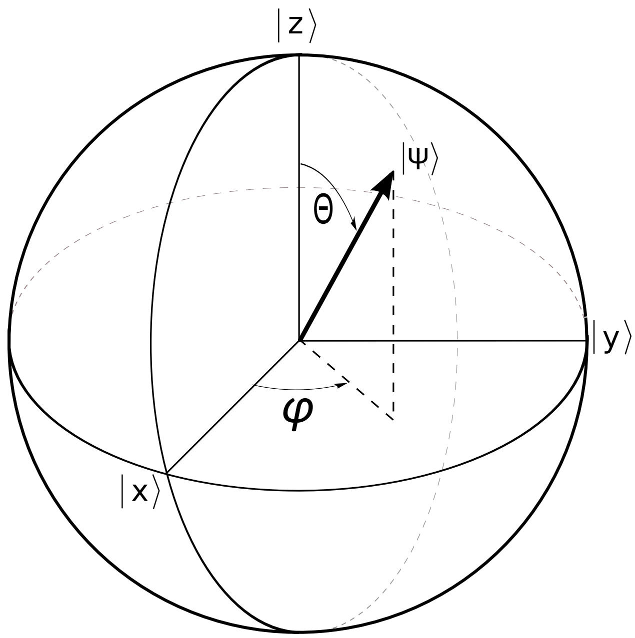

Figure 10.1: Representation of the state of a spin$-\frac{1}{2}$ particle as a point on the surface of the Bloch Sphere.

Given our generalized expression for a quantum state on the Bloch sphere ($|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$), we see that the action of the Hadamard gate is a π rotatation about the $\pi/4$ axis in the $x-z$-plane:y-axis.

Spatial Interpretation of Spin and Pauli Spin Matrices

Let's return to the question about what we mean spatially when we say that the electron is in the spin up state $|\uparrow\rangle$. We can understand this by referring to our Bloch sphere picture of the spin qubit. Let us align the z-axis with the direction "up" associated with spin up (say, as defined by an external B-field). Then the spin up state is identified with the positive z direction or the $|0\rangle$ state on the Bloch sphere. The spin down state is identified with the negative z direction or the $|1\rangle$ state on the Bloch sphere. The positive x direction corresponds to the state $1/\sqrt{2}\,|\uparrow\rangle + 1/\sqrt{2}\,|\downarrow\rangle$. Thus the Bloch sphere provides us a way of translating between the abstract vector space in which the state of the spin qubit resides and real 3-dimensional space.

Let us now consider how we would measure whether a spin qubit is in the state spin up $|\uparrow\rangle$ or spin down $|\downarrow\rangle$. We must define an operator whose eigenvectors are $|\uparrow\rangle$ and $|\downarrow\rangle$ (since only through such an operator can we associate a measurable quantity with spin-up or spin-down states). The operator that achieves this is the Pauli matix σ_z:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvectors of this operator are clearly $|0\rangle$ with eigenvalue 1 and $|1\rangle$ with eigenvalue -1.

Similarly if we wish to measure whether the spin points in the plus x direction ($|+\rangle$) or minus x direction ($|-\rangle$) on the Bloch sphere, we would use the Pauli matrix σ_x:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Its eigenvectors are $|+\rangle = 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle$ with eigenvalue 1 and $|-\rangle = 1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle$ with eigenvalue -1.

Finally if we wish to measure whether the spin points in the plus y direction or the minus y direction we would use the Pauli matrix σ_y:

$$\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The magnitude of the spin (intrinsic angular momentum) for an an electon is $\hbar/2$, and so the spin operators are the Pauli operators scaled by $\hbar/2$: $\hat{S}_x = \hbar/2\sigma_x$, $\hat{S}_y = \hbar/2\sigma_y$, $\hat{S}_z = \hbar/2\sigma_z$. These spin operators will play a central role in our study of spin.

Let us now step back and consider what we have learnt so far about spin. Spin is the intrinsic angular momentum carried by elementary particles, and for spin 1/2 particles such as electrons and protons spin is described by a qubit. But how do we reconcile this unit vector in an abstract vector space with the picture of an angular momentum vector pointing in some direction in real Euclidean space? We do this via the Bloch sphere, which maps qubit states onto points on the unit sphere in 3 dimensions.

There are several counter-intuitive consequences of what we have already described. Spin about the three axis are not independent. The z and y component of the spin completely determine the spin (and so do the z and $-z$ component!!). Indeed, we cannot independently measure the spin about the x, y and z axes. Another way of saying this is that the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ do not commute.

We express this via the commutation relations (verify these!) [1]:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

Finally, it will be useful for our later treatment to introduce a fourth operator $\hat{S}^2 = \hat{S}_x^{\,2} + \hat{S}_y^{\,2} + \hat{S}_z^{\,2}$. It is easy to verify that each of the Pauli spin matrices squares to the identity matrix, and therefore for the electron $\hat{S}^2 = 3\hbar^2/4I$. Our final commutation relation simply says that $\hat{S}^2$ commutes with each of the other three spin operators:

$[\hat{S}^2, \hat{S}_i] = 0$

History and a semi-classical picture

The history of the development of spin is an interesting one. In 1924 Pauli postulated a "two-valued quantum degree of freedom" (a qubit in our notation) associated with the electron in the outermost

[1] recall $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

shell, which helped him formulate the Pauli exclusion principle. But he sharply criticized Kronig's early suggestions that this degree of freedom was produced by a rotation of the electron. The "discovery" of spin is largely credited to two dutch physicists Uhlenbeck and Goudsmit who, in 1925, introduced it to explain the onset of new energy levels for hydrogen atoms in a magnetic field:

This can be explained if an electron behaves like a little magnet, if it has an *intrinsic* magnetic moment $\vec{\mu}$, since a magnetic moment in a magnetic field $\vec{B}$ has an energy $E = -\vec{\mu} \cdot \vec{B}$. In the context of QM, new energy levels come from $\vec{\mu}$ either parallel or anti-parallel to $\vec{B}$.

But where does $\vec{\mu}$ come from, and how do we explain its QM behavior?

The simplest explanation of the origin of $\vec{\mu}$ is "Classical": Classically, a loop of current gives rise to a magnetic moment $\vec{\mu}$:

The energy $E = -\vec{\mu} \cdot \vec{B}$ comes from $\vec{I} \times \vec{B}$ force of current in a B-field (Lorentz force). The lowest energy, and thereby the place where "the system wants to go", is obtained when the magnetic moment and B-field line up.

So a plausible explanation for "intrinsic" magnetic moment $\vec{\mu}$ of an electron is that the electron spins about some axis (thus effectively creating a loop of current about that axis). We can then express the magnetic moment $\vec{\mu}$ in terms of the angular momentum by a simple calculation:

Consider a charge of mass m moving in a circle with velocity v. Then its angular momentum is given by $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$. So in magnitude $L = mvr$. Also, its magnetic moment is given by:

$$\mu = (current)\,(Area) = \frac{e}{\tau} \cdot \pi r^2$$

But $\tau = \frac{2\pi r}{v}$, so $\mu = \frac{e}{2} \cdot vr$.

But now we can express the magnetic moment in terms of the angular momentum:

$$\mu = \frac{e}{2} \cdot vr = \frac{e}{2} \cdot \frac{L}{m} \Rightarrow \vec{\mu} = -\frac{e}{2m}\vec{L}$$

The central hypothesis of electron spin due to Uehlenbeck and Goudsmit states that each electron has an intrinsic angular momentum of spin $\vec{S}$ of magnitude $\hbar/2$ and with an associated magnetic moment $\vec{\mu} = -\frac{ge}{2m}\vec{S}$.

Compare this to the classical equation above: $\vec{\mu} = -\frac{e}{2m}\vec{L}$. What is g? g is called the g-factor and it is a unitless correction factor due to QM. For electrons, $g \approx 2$. For protons, $g \approx 5.6$. You should also note that $\frac{m_{proton}}{m_{electron}} \approx 2000$, so we conclude that $\vec{\mu_{proton}} \ll \vec{\mu_{electron}}$.

10.2 Stern-Gerlach Apparatus

A Stern-Gerlach device is simply a magnet set up to generate a particular inhomogeneous $\vec{B}$ field. When a particle with spin state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is shot through the apparatus from the left, its spin-up portion is deflected upward, and its spin-down portion downward. The particle's spin becomes entangled with its position! Placing detectors to intercept the outgoing paths therefore measures the particle's spin.

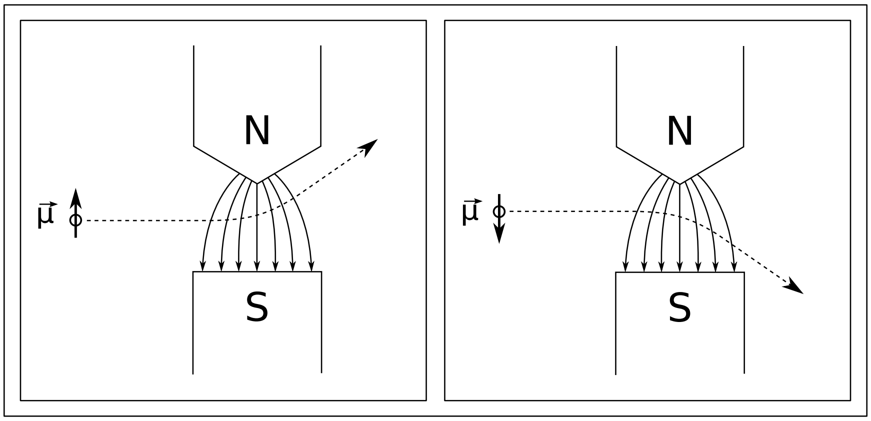

Figure 10.2: Stern-Gerlach device showing the deflection as a function of the orientation of the magnetic moment.

Why does this work? We'll give a semiclassical explanation – mixing the classical $\vec{F} = m\vec{a}$ and the quantum $H\psi = E\psi$ – which is quite wrong, but gives the correct intuition. [See Griffith's § 4.4.2, pp. 162-164 for a more complete argument.] Now the potential energy due to the spin interacting with the field is

$$E = -\vec{\mu} \cdot \vec{B}\ ,$$

so the associated force is

$$\vec{F}_{\text{spin}} = -\vec{\nabla} E = \vec{\nabla}(\vec{\mu} \cdot \vec{B})\ .$$

At the center $\vec{B} = B(z)\hat{z}$, with $\frac{\partial B}{\partial z} < 0$, so $\vec{F} = \vec{\nabla}(\mu_z B(z)) = \mu_z \frac{\partial B}{\partial z}\hat{z}$. The magnetic moment $\vec{\mu}$ is related to spin $\vec{S}$ by $\vec{\mu} = \frac{gq}{2m}\vec{S} = -\frac{e}{m}\vec{S}$ for an electron. Hence

$$\vec{F} = \tfrac{e}{m}\left|\tfrac{\partial B}{\partial z}\right| S_z \hat{z}\ ;$$

if the electron is spin up, the force is upward, and if the electron is spin down, the force is downward.

10.3 Initialize a Qubit

- How can we create a beam of qubits in the state $|\psi\rangle = |0\rangle$? Pass a beam of spin-$\frac{1}{2}$ particles with randomly oriented spins through a Stern-Gerlach apparatus oriented along the z axis. Intercept the downward-pointing beam, leaving the other beam of $|0\rangle$ qubits.

  Note that we *measure* the spin when we intercept an outgoing beam – after this measurement, the experiment is probabilistic and not unitary.

- How can we create a beam of qubits in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$? First find the point on the Bloch sphere corresponding to $|\psi\rangle$. That is, write

  $$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \cos\tfrac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}\,|1\rangle$$

(up to a phase), where

$$\tan\tfrac{\theta}{2} = \left|\tfrac{\beta}{\alpha}\right| \qquad e^{i\varphi} = \tfrac{\beta/|\beta|}{\alpha/|\alpha|} \ .$$

The polar coordinates θ, φ determine a unit vector $\hat{n} = \cos\varphi\sin\theta\hat{x} + \sin\varphi\sin\theta\hat{y} + \cos\theta\hat{z}$. Now just point the Stern-Gerlach device in the corresponding direction on the Bloch sphere, and intercept one of the two outgoing beams. That is, a Stern-Gerlach device pointed in direction $\hat{n}$ measures $S_{\hat{n}} = \hat{n} \cdot \hat{\vec{S}}$.

- How can we implement a unitary (deterministic) transformation? We need to evolve the wave function according to a Hamiltonian $\hat{H}$. Then

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi(0)\rangle$$

solves the Schrödinger equation (if $\hat{H}$ is time-independent). In the next lecture we will show how to accomplish an arbitrary single-qubit unitary gate (a rotation on the Bloch sphere) by applying a precise magnetic field for some precise amount of time: Larmor precession.