Quantum Mechanics & Quantum Computation

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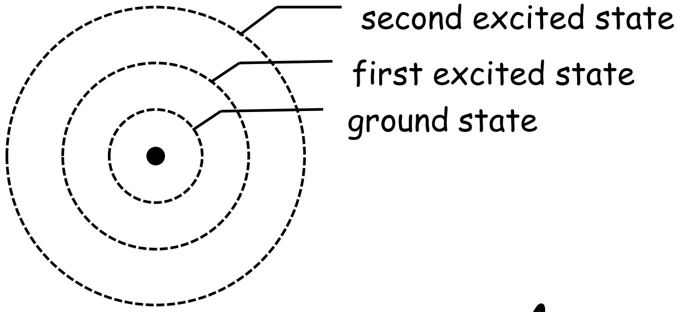
Lecture 3: Axioms of QM, two Qubits, Entanglement

K-level systems, bra-ket notation

K-level system:

$|0\rangle, |1\rangle, \ldots, |K-1\rangle$

Energy of an electron in an atom



Superposition Principle:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{K-1}|K-1\rangle$$

$$\alpha_j \in \mathbb{C}$$

$$\sum_{j=0}^{K-1} |\alpha_j|^2 = 1$$

Measurement Axiom:

$$P[j] = |\alpha_j|^2$$

New State $= |\psi'\rangle = |j\rangle$

$$|\psi\rangle = \left(\tfrac{1}{2} + \tfrac{i}{2}\right)|0\rangle - \tfrac{1}{2}|1\rangle + \tfrac{i}{2}|2\rangle$$

Measure:

$P[0] = \frac{1}{2}$ $\quad |\psi'\rangle = |0\rangle$

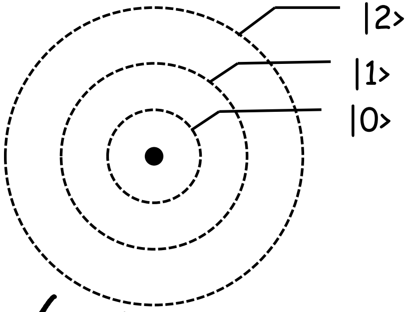
$P[1] = \frac{1}{4}$ $\quad |\psi'\rangle = |1\rangle$

$P[2] = \frac{1}{4}$ $\quad |\psi'\rangle = |2\rangle$.

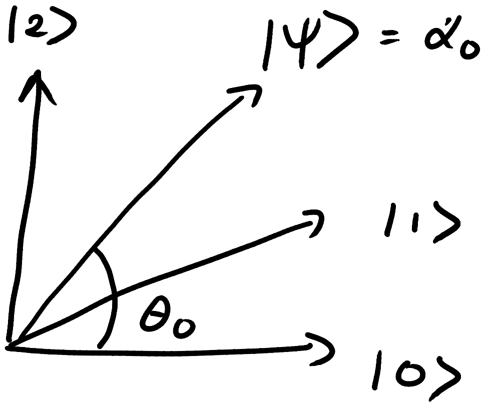
K-level system, Geometric Interpretation:

Superposition Principle: $|\psi\rangle \in \mathbb{C}^K$

State is a unit vector in a Hilbert space $\mathbb{C}^K$.



$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle = \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix} \in \mathbb{C}^3$$



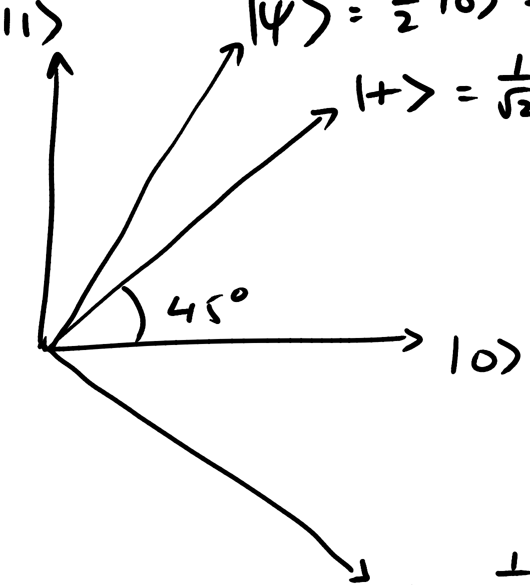
$$P[0] = \cos^2\theta_0$$

$$|\psi\rangle = \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix}$$

inner product $= \bar{\alpha}_0\beta_0 + \bar{\alpha}_1\beta_1 + \bar{\alpha}_2\beta_2$

$= (\bar{\alpha}_0\ \ \bar{\alpha}_1\ \ \bar{\alpha}_2)\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix}$ $\qquad \cos\theta = |\bar{\alpha}_0\beta_0 + \bar{\alpha}_1\beta_1 + \bar{\alpha}_2\beta_2|$

Example: measurement in sign basis



$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\boxed{P[-] = \frac{2-\sqrt{3}}{4}}$$

$$P[+] = \left[\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right]^2$$

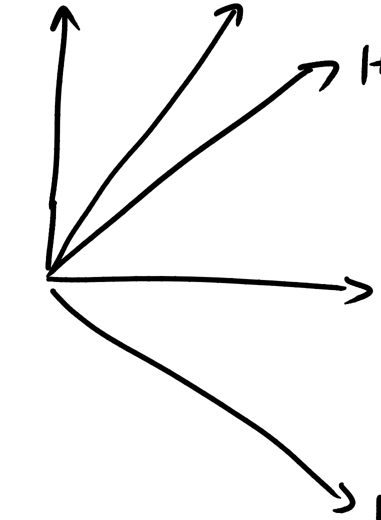
$$= \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right)^2$$

$$= \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2$$

$$= \frac{1+3+2\sqrt{3}}{8}$$

$$= \frac{2+\sqrt{3}}{4}$$

Example: measurement in sign basis



$|1\rangle$

$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|0\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

$$|\psi\rangle = \frac{1}{2}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right)$$

$$= \underbrace{\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)}|+\rangle + \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)|-\rangle$$

$$P[+] = \frac{1+3+2\sqrt{3}}{8} = \frac{2+\sqrt{3}}{4}$$

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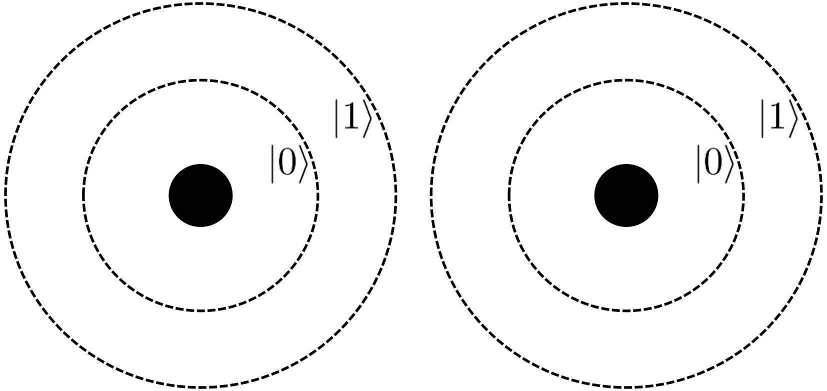
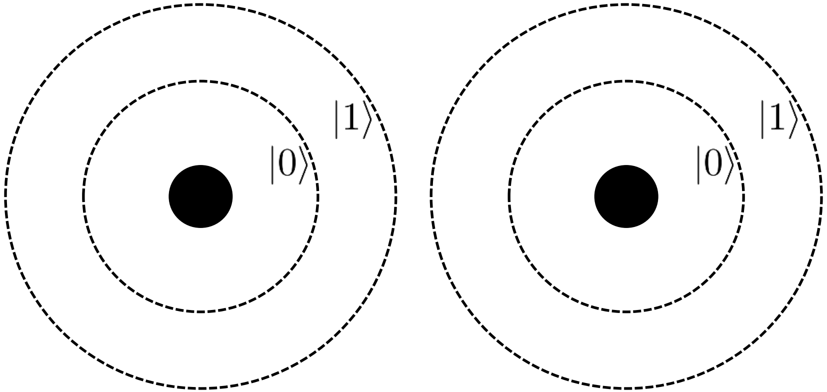
Two Qubits

Two Qubits



Partial Measurement



- What is the result of measuring just the first qubit?

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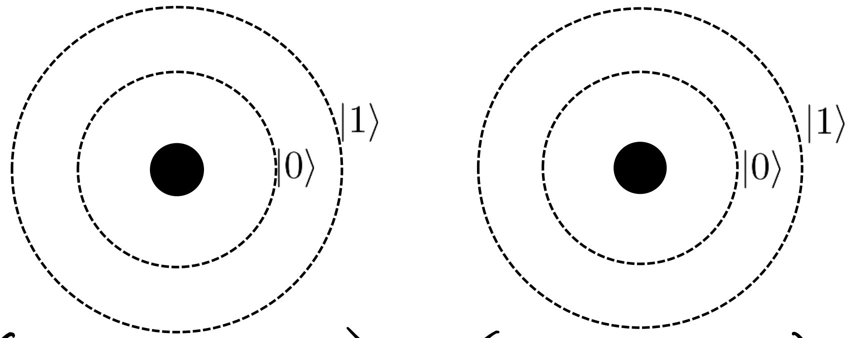
Entanglement

Composite System



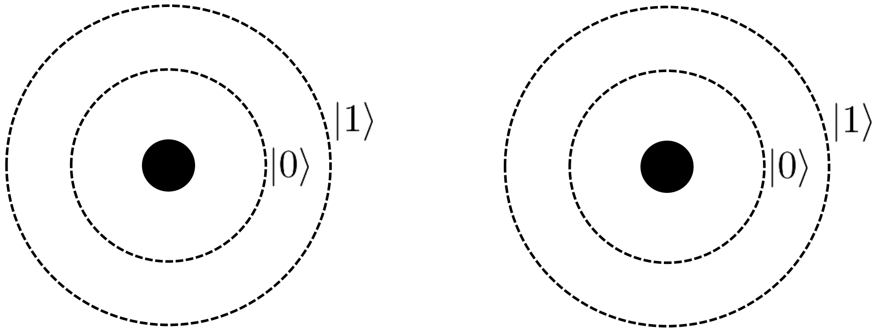
$$\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right) \otimes \left(\beta_0|0\rangle + \beta_1|1\rangle\right)$$

$$\underset{\alpha_0\beta_0}{\underset{\shortparallel}{a_{00}}}|00\rangle + \underset{\alpha_0\beta_1}{\underset{\shortparallel}{a_{01}}}|01\rangle + \underset{\alpha_1\beta_0}{\underset{\shortparallel}{a_{10}}}|10\rangle + \underset{\alpha_1\beta_1}{\underset{\shortparallel}{a_{11}}}|11\rangle$$

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)$$

$$\frac{1}{2\sqrt{2}}|00\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$$

Bell State



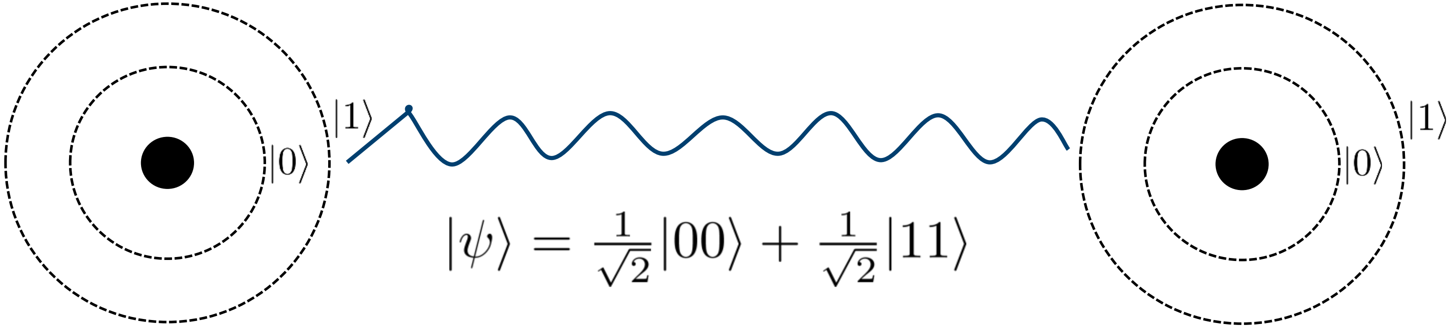
$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$(\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$$

$$\underbrace{\alpha_0\beta_0}_{=\frac{1}{\sqrt{2}}}|00\rangle + \underbrace{\alpha_0\beta_1}_{=0}|01\rangle + \underbrace{\alpha_1\beta_0}_{=0}|10\rangle + \underbrace{\alpha_1\beta_1}_{=\frac{1}{\sqrt{2}}}|11\rangle$$

$$\alpha_0, \beta_0, \alpha_1, \beta_1 \neq 0 \quad \times$$

Measuring the Bell State



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Measure : $P[00] = \frac{1}{2}$

$P[11] = \frac{1}{2}$.

$P[0] = \frac{1}{2}$ New state = $|00\rangle$ ← $P[0] = \frac{1}{2}$

$P[1] = \frac{1}{2}$ New state = $|11\rangle$ ← $P[1] = \frac{1}{2}$

Covalent bond : <u>spin</u> $\alpha_0|\uparrow\rangle + \alpha_1|\downarrow\rangle$

$\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$ $\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

“I would not call [entanglement] *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.”

Erwin Schrödinger (1935)

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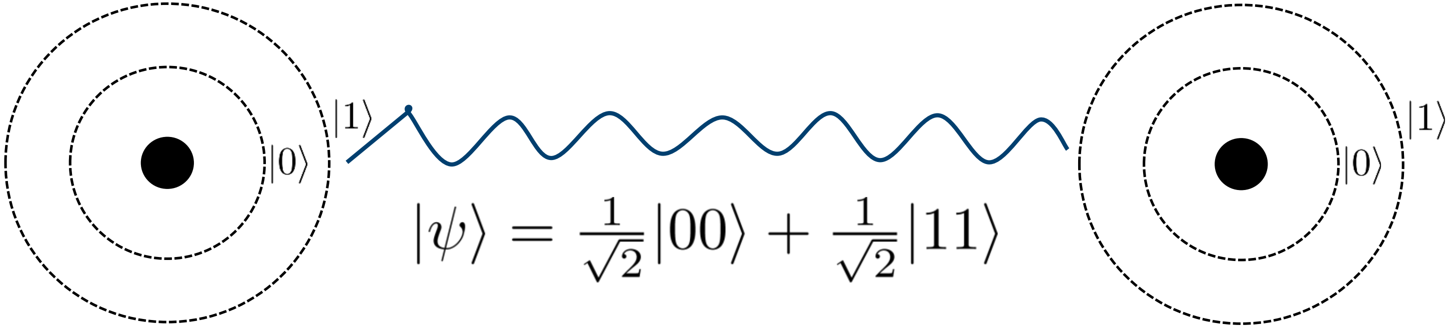
EPR Paradox

Measuring Bell State in sign basis



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

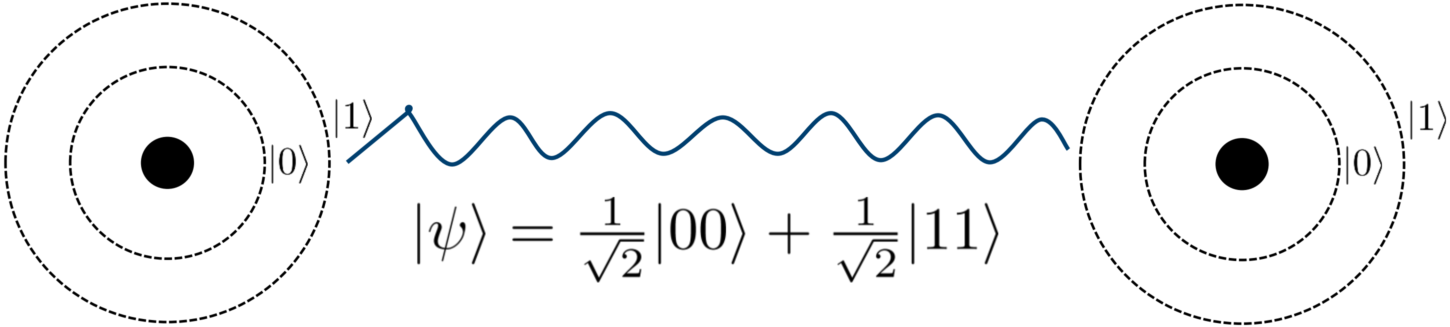
Sign Basis

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$P[+] \longrightarrow P[+]$?

Re-writing Bell State in sign basis



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$P[+] = \frac{1}{2}$ new state $= |++\rangle \longrightarrow P[+] = 1$

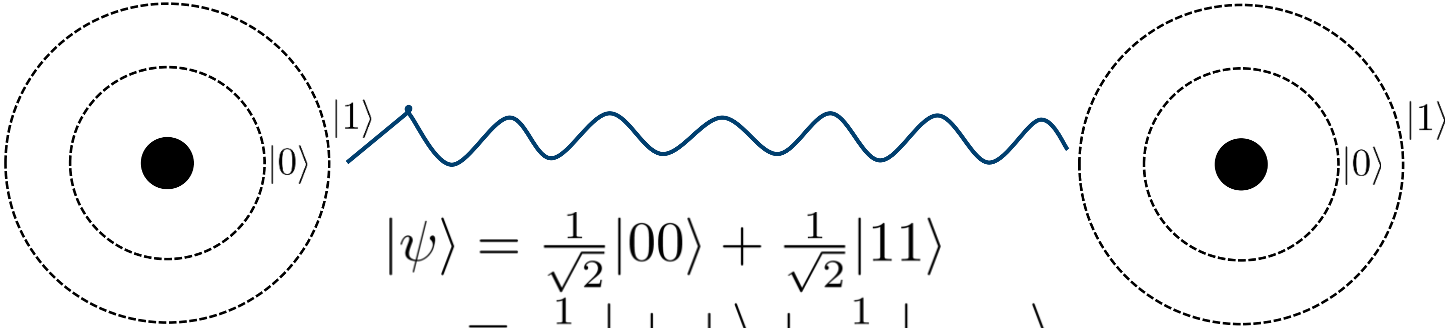
$P[-] = \frac{1}{2}$ $|--\rangle \longrightarrow P[-] = 1$

Claim:
$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$= \frac{1}{\sqrt{2}}\underline{|++\rangle} + \frac{1}{\sqrt{2}}|--\rangle$$

$$\frac{1}{\sqrt{2}}\left[\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) + \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\left(\tfrac{1}{2}|00\rangle + \tfrac{1}{2}\cancel{|01\rangle} + \tfrac{1}{2}\cancel{|10\rangle} + \tfrac{1}{2}|11\rangle\right) + \left(\tfrac{1}{2}|00\rangle - \tfrac{1}{2}\cancel{|01\rangle} - \tfrac{1}{2}\cancel{|10\rangle} + \tfrac{1}{2}|11\rangle\right)\right]$$

Einstein, Podolsky, Rosen (EPR) Paradox (1935)



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

Bit
Sign

Measure Sign

+

Measure Bit 0

Local realism

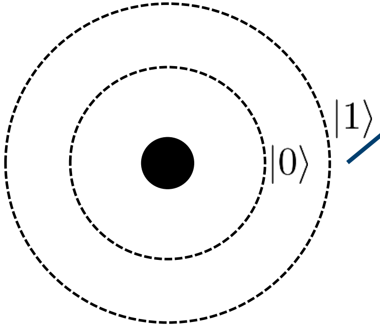
It is inconceivable that inanimate Matter should, without the mediation of something else, which is not material, operate upon, and affect other matter without mutual contact.

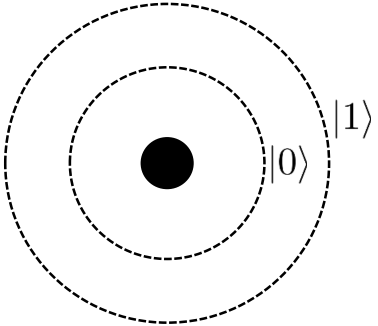
Issac Newton

I think that matter must have a separate reality independent of the measurements. That is an electron has spin, location and so forth even when it is not being measured. I like to think that the moon is there even if I am not looking at it.

Albert Einstein

Einstein, Podolsky, Rosen (EPR) Paradox (1935)





$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

Measure Bit : 0

Measure sign : +

New state $|00\rangle$

$$= (|0\rangle)(|0\rangle)$$