Chapter 2

Entanglement

What are the allowable quantum states of systems of several particles? The answer to this is enshrined in the addendum to the first postulate of quantum mechanics: the superposition principle. In this chapter we will consider a special case, systems of two qubits. In keeping with our philosophy, we will first approach this subject naively, without the formalism of the formal postulate. This will facilitate an intuitive understanding of the phenomenon of quantum entanglement — a phenomenon which is responsible for much of the "quantum weirdness" that makes quantum mechanics so counter-intuitive and fascinating.

2.1 Two qubits

Now let us examine a system of two qubits. Consider the two electrons in two hydrogen atoms, each regarded as a 2-state quantum system:

Since each electron can be in either of the ground or excited state, classically the two electrons are in one of four states – 00, 01, 10, or 11 – and represent 2 bits of classical information. By the superposition principle, the quantum state of the two electrons can be any linear combination of these four classical states:

$$|\psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle \quad ,$$

where $\alpha_{ij} \leq \mathbb{C}$, $\sum_{ij} |\alpha_{ij}|^2 = 1$. Of course, this is just Dirac notation for the unit vector in $\mathbb{C}^4$:

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

Measurement

As in the case of a single qubit, even though the state of two qubits is specified by four complex numbers, most of this information is not accessible by measurement. In fact, a measurement of a two qubit system can only reveal two bits of information. The probability that the outcome of the measurement is the two bit string $x \in \{0,1\}^2$ is $|\alpha_x|^2$. Moreover, following the measurement the state of the two qubits is $|x\rangle$. i.e. if the first bit of x is j and the second bit k, then following the measurement, the state of the first qubit is $|j\rangle$ and the state of the second is $|k\rangle$.

An interesting question comes up here: what if we measure just the first qubit? What is the probability that the outcome is 0? This is simple. It is exactly the same as it would have been if we had measured both qubits: $\Pr\{\text{1st bit } = 0\} = \Pr\{00\} + \Pr\{01\} = |\alpha_{00}|^2 + |\alpha_{01}|^2$. Ok, but how does this partial measurement disturb the state of the system?

The answer is obtained by an elegant generalization of our previous rule for obtaining the new state after a measurement. The new superposition is obtained by crossing out all those terms of $|\psi\rangle$ that are inconsistent with the outcome of the measurement (i.e. those whose first bit is 1). Of course, the sum of the squared amplitudes is no longer 1, so we must renormalize to obtain a unit vector:

$$|\phi_{\text{new}}\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

Entanglement

Suppose the first qubit is in the state $3/5\,|0\rangle + 4/5\,|1\rangle$ and the second qubit is in the state $1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle$, then the joint state of the two qubits is $(3/5\,|0\rangle + 4/5\,|1\rangle)(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle) = 3/5\sqrt{2}\,|00\rangle - 3/5\sqrt{2}\,|01\rangle + 4/5\sqrt{2}\,|10\rangle - 4/5\sqrt{2}\,|11\rangle$.

More generally, if the state of the first qubit is $\alpha_0|0\rangle + \alpha_1|1\rangle$ and the state of the second qubit is $\beta_0|0\rangle + \beta_1|1\rangle$, then the joint state of the two qubits is $\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$.

Can every state of two qubits be decomposed in this way? Our classical intuition would suggest that the answer is obviously affirmative. After all each of the two qubits must be in some state $\alpha|0\rangle + \beta|1\rangle$, and so the state of the two qubits must be the product. In fact, there are states such as $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ which cannot be decomposed in this way as a state of the first qubit and that of the second qubit. Can you see why? Such a state is called an entangled state. When the two qubits are entangled, we cannot determine the state of each qubit separately. The state of the qubits has as much to do with the relationship of the two qubits as it does with their individual states.

If the first (resp. second) qubit of $|\Phi^+\rangle$ is measured then the outcome is 0 with probability 1/2 and 1 with probability 1/2. However if the outcome is 0, then a measurement of the second qubit results in 0 with certainty. This is true no matter how large the spatial separation between the two particles.

The state $|\Phi^+\rangle$, which is one of the Bell basis states, has a property which is even more strange and wonderful. The particular correlation between the measurement outcomes on the two qubits holds true no matter which rotated basis a rotated basis $|v\rangle, |v^\perp\rangle$ the two qubits are measured in, where $|0\rangle = \alpha|v\rangle + \beta|v^\perp\rangle$ and $|1\rangle = -\beta|v\rangle + \alpha|v^\perp\rangle$. This can bee seen as,

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\left(\alpha|v\rangle + \beta\left|v^\perp\right\rangle\right) \otimes \left(\alpha|v\rangle + \beta\left|v^\perp\right\rangle\right)\right) \\
&\quad - \frac{1}{\sqrt{2}}\left(\left(-\beta|v\rangle + \alpha\left|v^\perp\right\rangle\right) \otimes \left(-\beta|v\rangle + \alpha\left|v^\perp\right\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(\left(\alpha^2 + \beta^2\right)|vv\rangle + \left(\alpha^2 + \beta^2\right)\left|v^\perp v^\perp\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|vv\rangle + |v^\perp v^\perp\rangle\right)
\end{aligned}$$

EPR Paradox:

Everyone has heard Einstein's famous quote "God does not play dice with the Universe". The quote is a summary of the following passage from Einstein's 1926 letter to Max Born: "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the Old One. I, at any rate, am convinced that He does not throw dice." Even to the end of his life, Einstein held on to the view that quantum physics is an incomplete theory and that some day we would learn a more complete and satisfactory theory that describes nature.

In what sense did Einstein consider quantum mechanics to be incomplete? Think about flipping a coin. For all common purposes, the outcome of a coin toss is random — heads half the time, and tails the other half. And this lines up exactly with our observations, but we know that randomness isn't the whole story. A more complete theory would say that if we knew *all* of the initial conditions of the coin *exactly* (position, momentum), then we could use Newton's laws of classical physics to figure out exactly how the coin would land, and therefore the outcome of the coin flip. Another way to say this is that the coin flip amplifies our lack of knowledge about the state of the system, and makes the outcome seem completely random. In the same way, Einstein believed that the randomness of quantum measurements reflected our lack of knowledge about additional degrees of freedom, or "hidden variables," of the quantum system.

Einstein sharpened this line of reasoning in a paper he wrote with Podolsky and Rosen in 1935, where they introduced the famous Bell states. The EPR argument works like this. For Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, when you measure first qubit (in the bit basis), the second qubit is determined (in the bit basis). What is even more remarkable is that if you measure the first qubit in the sign basis, the second qubit is determined in the sign basis. You should verify that in the sign basis $(|+\rangle, |-\rangle)$, the state $|\Phi^+\rangle$ can be written as $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$. It follows that knowledge of the sign of one qubit completely determines the other.

Now lets suppose the qubits are very far apart, say one light-second. If we measure qubit 1 in the standard basis, then measure qubit 2 a half second later in the same basis, the two measurements must agree. Then qubit 2 must have been in a definite state for a half second before it was measured: from the instant we measured qubit 1, we knew qubit 2. But the qubits couldn't have communicated any information in that time.

What if we had measured the first qubit in the $|+\rangle, |-\rangle$ basis instead? Then similarly for half a second before we measure qubit 2, it was in a definite state in the $|+\rangle, |-\rangle$ basis. But qubit 2 could not possibly know which basis qubit 1 was measured in until a full second *after* we measure qubit 1! This is because we assumed that light takes a one second to travel from qubit 1 to qubit 2. This appears to contradict the uncertainty principle for $|\pm\rangle$ and $|0\rangle, ket1$ states says that there is no definite $|\pm\rangle$ state that is also a definite $|0\rangle, |1\rangle$ state.

Einstein, Podolsky, and Rosen concluded that since qubit 2 cannot have any information about which basis qubit 1 was measured in, its state in both bit and sign bases is simultaneously determined, something that quantum mechanics does not allow. EPR therefore suggested that quantum mechanics is an incomplete theory, and there is a more complete theory where "God does not throw dice." Until his death in 1955, Einstein tried to formulate a more complete "local hidden variable theory" that would describe the predictions of quantum mechanics, but without resorting to probabilistic outcomes.

But in 1964, almost three decades after the EPR paper, John Bell showed that properties of Bell (EPR) states were not merely fodder for a philosophical discussion, but had verifiable consequences: local hidden variables[1] are not the answer. He described an experiment to be performed on two qubits entangled in a Bell state such that a local hidden variable theory would disagree with quantum mechanics about the outcome. The Bell experiment has been performed to increasing accuracy, originally by Aspect, and the results have always been consistent with the predictions of quantum mechanics and inconsistent with local hidden variable theories.

2.2 Bell's Thought Experiment

Bell considered the following experiment: let us assume that two particles are produced in the Bell state $|\Phi^+\rangle$ in a laboratory, and the fly in opposite directions to two distant laboratories. Upon arrival, each of the two qubits is subject to one of two measurements. The decision about which of the two experiments is to be performed at each lab is made randomly at the last moment, so that speed of light considerations rule out information about the choice at one lab being transmitted to the other. The measurements are cleverly chosen to distinguish between the predictions of quantum mechanics and any local hidden variable theory. Concretely, the experiment measures the correlation between the outcomes of the two experiments. The choice of measurements is such that any classical hidden variable theory predicts that the correlation between the two outcomes can be at most 0.75, whereas quantum mechanics predicts that the correlation is $\cos^2 \pi/8 \approx 0.85$. Thus the experiment allows us to distinguish between the predictions of quantum mechanics and any local hidden variable theory! We now describe the experiment in more detail.

The two experimenters A and B (for Alice and Bob) each receives one qubit of a Bell state $|\Phi^+\rangle$, and measures it in one of two bases depending upon the value of a random bit r_A and r_B respectively. Denote by a and b respectively the outcomes of the measurements. We are interested in the highest achievable correlation between the two quantities $r_A \times r_B$ and $a + b(mod 2)$. We will see below that there is a particular choice of bases for the quantum measurements made by A and B such that $P[r_A \times r_B = a + b(mod 2)] = \cos^2 \pi/8 \approx .85$. Before we do so, let us see why no classical hidden variable theory allows a correlation of over 0.75. i.e. $P[r_A \times r_B = a + b(mod 2)] \leq 0.75$.

[1] We will describe what we mean by a local hidden variable theory below after we start describing the actual experiment

We can no longer postpone a discussion about what a local hidden variable theory is. Let us do so in the context of the Bell experiment. In a local hidden variable theory, when the Bell state was created, the two particles might share an arbitrary amount of classical information, x. This information could help them coordinate their responses to any measurements they are subjected to in the future. By design, the Bell experiment selects the random bits r_A an r_B only after the two particles are too far apart to exchange any further information before they are measured. Thus we are in the setting, where A and B share some arbitrary classical information x, and are given as input independent, random bits x_A an x_B as input, and must output bits a and b respectively to maximize their chance of achieving $r_A \times r_B = a + b(mod2)$. It can be shown that the shared information x is of no use in increasing this correlation, and indeed, the best they can do is to always output $a = b = 0$. This gives $P[r_A \times r_B = a + b(mod2)] \leq .75$.

Let us now describe the quantum measurements that achieve greater correlation. They are remarkably simple to describe:

- if $r_A = 0$, then Alice measures in the standard $|0\rangle$ / $|1\rangle$ basis.
- if $r_A = 1$, then Alice measures in the $\pi/4$ basis (i.e. standard basis rotated by $\pi/4$).
- if $r_B = 0$, then Bob measures in the $\pi/8$ basis.
- if $r_B = 1$, then Bob measures in the $-\pi/8$ basis.

The analysis of the success probability of this experiment is also beautifully simple. We will show that in each of the four cases $r_A = r_B = 0$, etc, the success probability $P[r_A \times r_B = a + b(mod2)] = \cos^2 \pi/8$.

We first note that if Alice and Bob measure in bases that make an angle θ with each other, then the chance that their measurement outcomes are the same (bit) is exactly $\cos^2 \theta$. This follows from the rotational invariance of $|\Phi^+\rangle$ and the following observation: if the first qubit is measured in the standard basis, then the outcome is outcome is an unbiased bit. Moreover the state of the second qubit is exactly equal to the outcome of the measurement — $|0\rangle$ if the measurement outcome is 0, say. But now if the second qubit is measured in a basis rotated by θ, then the probability that the outcome is also 0 is exactly $\cos^2 \theta$.

Now observe that in three of the four cases, where $x_A \cdot x_B = 0$, Alice and Bob measure in bases that make an angle of $\pi/8$ with each other. By our observation above, $P[a + b \equiv 0 \mod 2] = P[a = b] = \cos^2 \pi/8$.

In the last case $x_A \cdot x_B = 1$, and they measure in bases that make an angle of $3\pi/8$ with each other. So, $P[a + b \equiv 0 \mod 2] = P[a \neq b] = \cos^2 3\pi/8 = \sin^2(\pi/2 - 3\pi/8) = sin^2\pi/8$. Therefore, $P[a + b \equiv 1 \mod 2] = 1 - \sin^2 \pi/8 = \cos^2 \pi/8$. So in each of the four cases, the chance that Alice and Bob succeed is $\cos^2 \pi/8 \approx .85$