Quantum Mechanics & Quantum Computation

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Lecture 5: Quantum Gates

Time evolution of a quantum system

Axioms of quantum mechanics

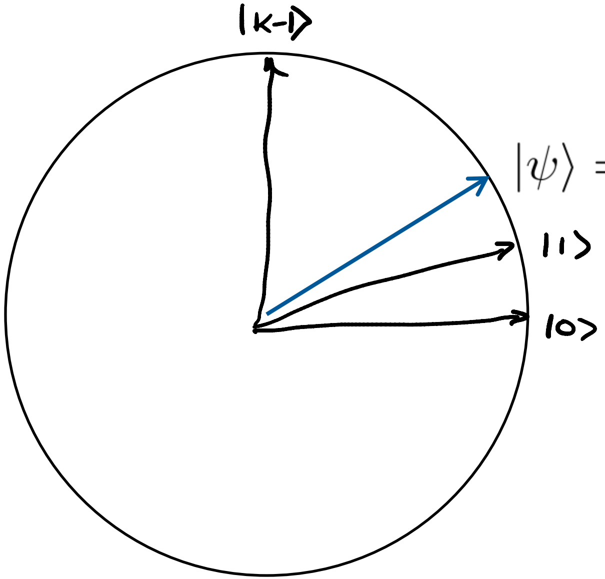
Axiom 1. Superposition principle

Axiom 2. Measurement

Axiom 3. Unitary evolution

Superposition

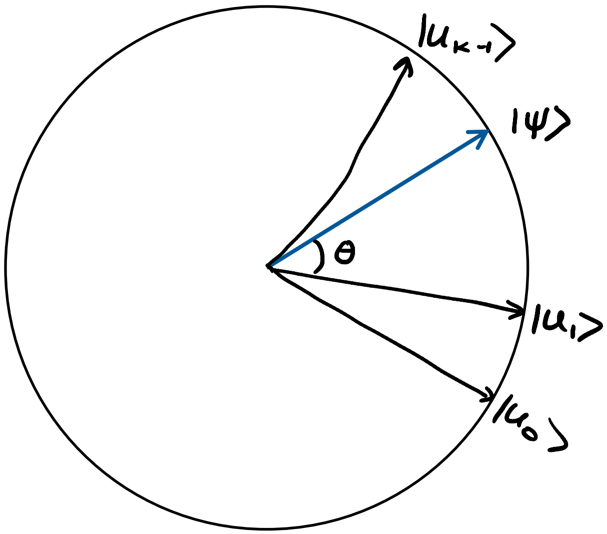
- Allowable states of k-level system: unit vector in a k-dimensional complex vector space (called a Hilbert space).



$$|\psi\rangle = \alpha_0|0\rangle + \cdots + \alpha_{k-1}|k-1\rangle = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \mathbb{C}^k$$

Measurement

- A measurement is specified by choosing an orthonormal basis.
- The probability of each outcome is the square of the length of the projection onto the corresponding basis vector.
- The state collapses to the observed basis vector.



New State

$$|\psi'\rangle = |u_j\rangle$$

$$j: \quad |(|u_j\rangle, |\psi\rangle)|^2$$

$$|\psi\rangle = \alpha_0|0\rangle + \cdots\cdots + \alpha_{k-1}|k-1\rangle$$

$$|u_j\rangle = \beta_0|0\rangle + \cdots\cdots + \beta_{k-1}|k-1\rangle$$

$$(|u_j\rangle, |\psi\rangle) = \bar{\beta}_0\alpha_0 + \cdots + \bar{\beta}_{k-1}\alpha_{k-1}$$

$$= \underbrace{(\bar{\beta}_0 \ \bar{\beta}_1 \cdots\cdots \bar{\beta}_{k-1})}\begin{pmatrix}\alpha_0\\ \vdots \\ \alpha_{k-1}\end{pmatrix}$$

Bra-ket notation

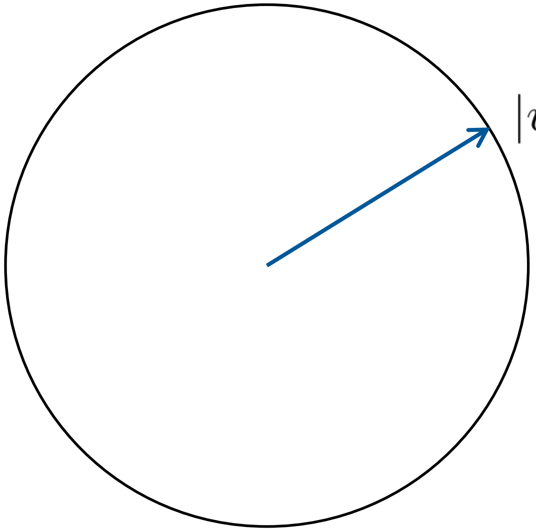
$$\langle u_j| = (\bar{\beta}_0 \ \bar{\beta}_1 \ \cdots \ \bar{\beta}_{k-1})$$

$$\text{inner product} = \langle u_j|\psi\rangle$$

$$\Pr[j] = |\langle u_j|\psi\rangle|^2$$

Evolution

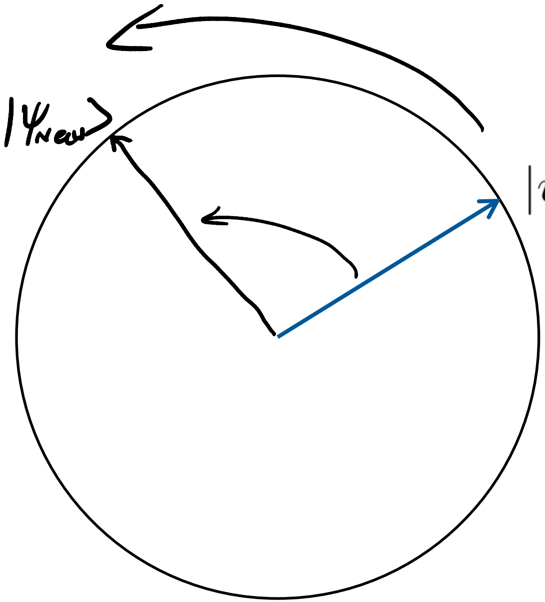
- How does the state evolve in time?



$$|\psi\rangle = \alpha_0|0\rangle + \cdots + \alpha_{k-1}|k-1\rangle = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \mathbb{C}^k$$

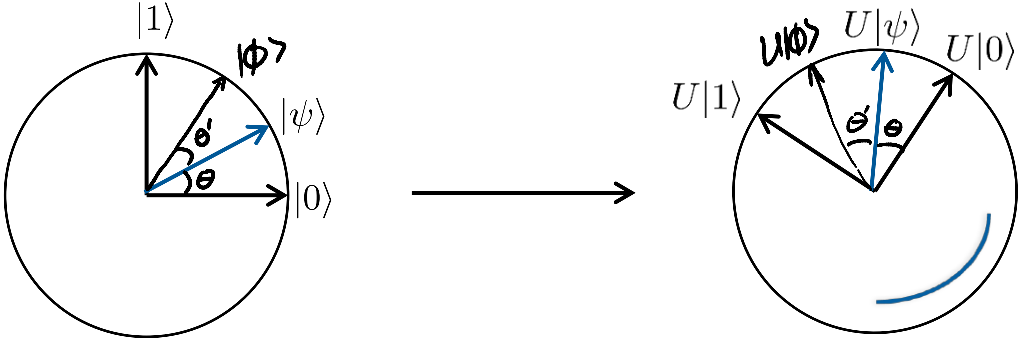
Evolution

- How does the state evolve in time? By a rotation of the Hilbert space!



$$|\psi\rangle = \alpha_0|0\rangle + \cdots + \alpha_{k-1}|k-1\rangle = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \mathbb{C}^k$$

Example: evolution of a qubit.

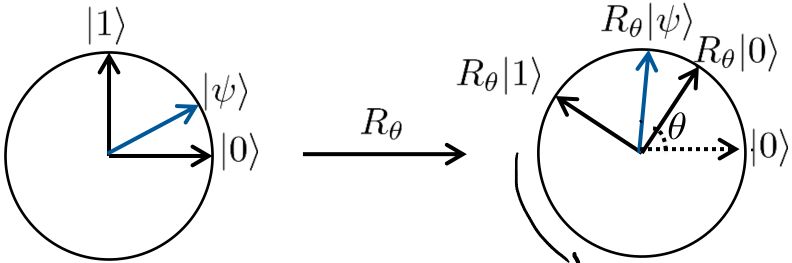


It is a rigid body rotation, meaning that the angles between vectors are preserved.

Rotation Matrix

- Rotation of the space is a linear transformation. Represent by a matrix:

- Example



$\begin{pmatrix}1\\0\end{pmatrix} = |0\rangle \xrightarrow{R_\theta} \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

$|1\rangle \xrightarrow{R_\theta} -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta^T$

$R_\theta R_{-\theta} = R_{-\theta} R_\theta = I$

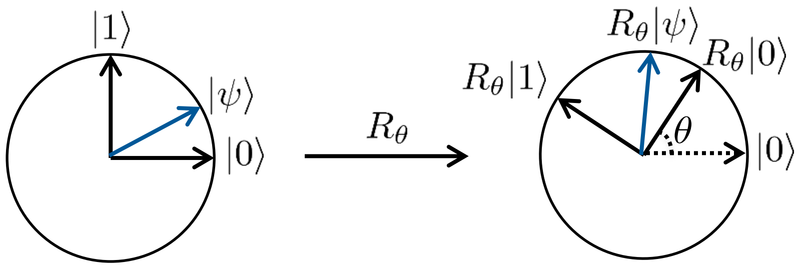
$\boxed{R_\theta R_\theta^T = R_\theta^T R_\theta = I}$

unitary.

Rotation Matrix

- Rotation of the space is a linear transformation. Represent by a matrix:

- Example



$$|0\rangle \longrightarrow \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

$$|1\rangle \longrightarrow -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow$$

$$\alpha(\cos\theta\,|0\rangle + \sin\theta\,|1\rangle)$$

$$+\beta(-\sin\theta\,|0\rangle + \cos\theta\,|1\rangle)$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ \end{pmatrix}$$

$$= (\alpha\cos\theta - \beta\sin\theta)\,|0\rangle + (\alpha\sin\theta + \beta\cos\theta)\,|1\rangle$$

$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta^T$$

$$R_\theta R_{-\theta} = I$$

$$R_\theta R_\theta^T = R_\theta^T R^\theta = I$$

Rotation Matrix

- Rotation of the space is a linear transformation. Represent by a matrix:

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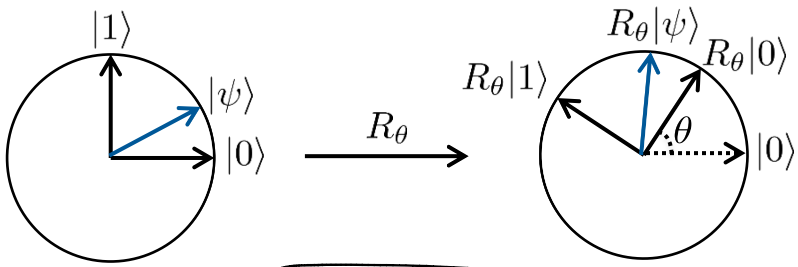

$$|0\rangle \longrightarrow \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

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$$\alpha|0\rangle + \beta|1\rangle \longrightarrow$$

$$\alpha(\cos\theta\,|0\rangle + \sin\theta\,|1\rangle)$$

$$+\beta(-\sin\theta\,|0\rangle + \cos\theta\,|1\rangle)$$

$$R_\theta = \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\alpha\cos\theta - \beta\sin\theta)\,|0\rangle + (\alpha\sin\theta + \beta\cos\theta)\,|1\rangle$$

$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta^T$$

$$R_\theta R_{-\theta} = I$$

$$R_\theta R_\theta^T = R_\theta^T R^\theta = I$$

Unitary Transformations

$\underline{\underline{\mathbb{C}}}^K$

$$\begin{array}{c} \\ U = \end{array} \begin{array}{cccc} |0\rangle & |1\rangle & & |K-1\rangle \\ \downarrow & \downarrow & & \downarrow \end{array}$$

$$U = \begin{pmatrix} \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ \cdot & \cdot & \cdots & \cdot \end{pmatrix}$$

U unitary $\Longleftrightarrow$

$$U^\dagger U = U U^\dagger = I$$

$$\underbrace{\begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \end{pmatrix}}_{U^\dagger} \underbrace{\begin{pmatrix} | & | & | \end{pmatrix}}_{U} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$U|0\rangle = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$$

$$U|1\rangle = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = c|0\rangle + d|1\rangle$$

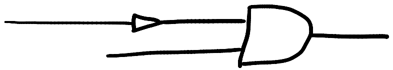
U preserves inner products:

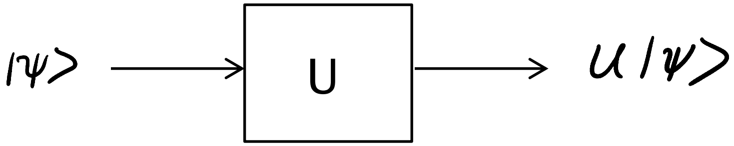
$|\phi\rangle$ & $|\psi\rangle$ $\qquad \langle\phi|\psi\rangle$

$U|\phi\rangle$ & $U|\psi\rangle$ $\qquad \langle\phi|U^\dagger U|\psi\rangle$

$\langle\phi|U^\dagger \;=$

Quantum gates





wire $\longleftrightarrow$ qubit

"Bit flip"

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{X} \longrightarrow \alpha_1|0\rangle + \alpha_0|1\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$X^\dagger X = X X^\dagger = I = X^2$$

"Phase flip"

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{Z} \longrightarrow \alpha_0|0\rangle - \alpha_1|1\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle$$

$$|+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

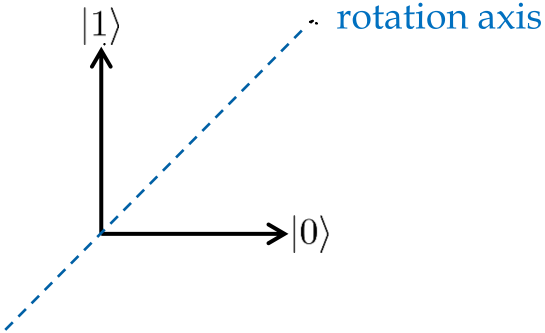
$$Z|+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle = |-\rangle$$

$$Z|-\rangle = |+\rangle$$

$$Z^\dagger Z = Z Z^\dagger = I = Z^2$$

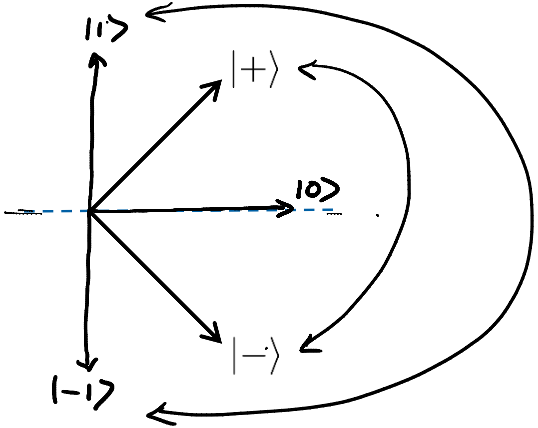
Bit flip

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Phase flip

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



"Hadamard transform"

$$\longrightarrow \boxed{H} \longrightarrow$$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|0\rangle \longrightarrow H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle$$

$$|1\rangle \longrightarrow H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle$$

$$H^{+}H = HH^{+} = I = H^{2}$$

$$|0\rangle \longleftrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|1\rangle \longleftrightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

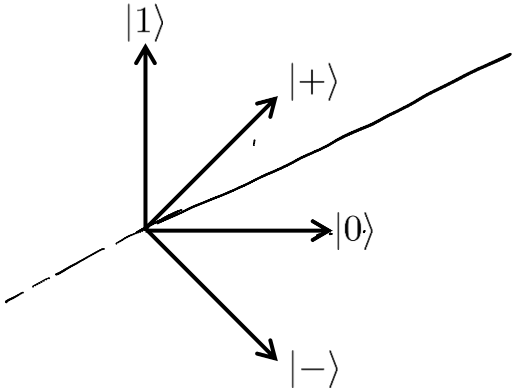
"Hadamard transform"

$$\longrightarrow \boxed{H} \longrightarrow$$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H|0\rangle = |+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = |-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$



"Bit flip"

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{X} \longrightarrow \alpha_1|0\rangle + \alpha_0|1\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

"Phase flip"

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{Z} \longrightarrow \alpha_0|0\rangle - \alpha_1|1\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

"Hadamard transform"

$$\longrightarrow \boxed{H} \longrightarrow$$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$X = HZH$

$H^2 = I$

$H = H^{-1}$

$$\begin{tikzcd}
\underline{|0\rangle} \arrow[r, dash, "X"] \arrow[d, dash, "H"'] & \underline{|1\rangle} \arrow[d, dash, "H"] \\
\underline{|+\rangle} \arrow[r, dash, "Z"'] & \underline{|-\rangle}
\end{tikzcd}$$

$$\begin{tikzcd}
|0\rangle \arrow[r, "H"] \arrow[rrr, bend right, leftrightarrow, "X"'] & |+\rangle \arrow[r, "Z"] & |-\rangle \arrow[r, "H"] & |1\rangle
\end{tikzcd}$$

- One-qubit gates:

$$\alpha_0|0\rangle+\alpha_1|1\rangle \longrightarrow \boxed{\mathrm{U}} \longrightarrow \alpha_0'|0\rangle+\alpha_1'|1\rangle$$

$$U=\begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad U^+=\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}$$

$$U^+U=UU^+=I \qquad \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}=\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$$

- What is the dimension of two-qubit gates?

$$\alpha_{00}|00\rangle+\alpha_{01}|01\rangle+\alpha_{10}|10\rangle+\alpha_{11}|11\rangle \;=\!\!\boxed{U}\!\!=\; \alpha'_{00}|00\rangle+\alpha'_{01}|01\rangle+\alpha'_{10}|10\rangle+\alpha'_{11}|11\rangle$$

$$U=\begin{pmatrix} a & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \alpha'_{00} \\ \alpha'_{01} \\ \alpha'_{10} \\ \alpha'_{11} \end{pmatrix}=U\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

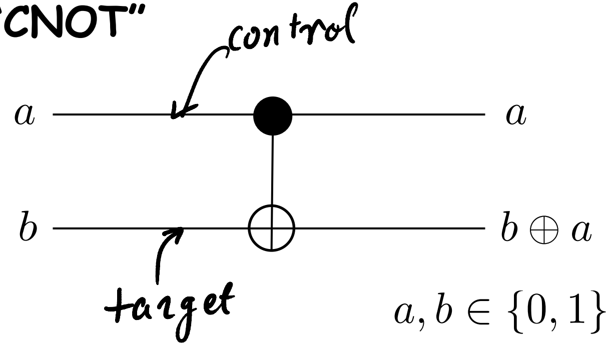
$$U|00\rangle = a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle$$

$$|01\rangle$$

U is a 4x4 unitary matrix.

$$UU^\dagger=U^\dagger U=I$$

"CNOT"



$a, b \in \{0, 1\}$

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

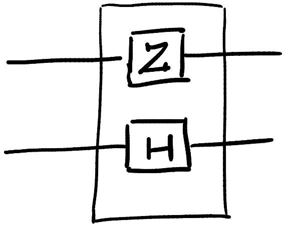
$$\Big\downarrow \text{CNOT}$$

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{11}|10\rangle + \alpha_{10}|11\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$CNOT^{+} = CNOT$$

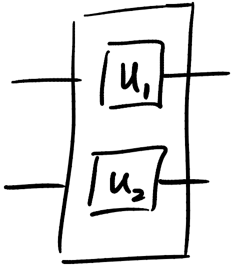
$$CNOT^{+} \cdot CNOT = CNOT^{2} = I$$



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \multicolumn{2}{c}{0} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & \\ \multicolumn{2}{c}{0} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



$$U_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$U_2 = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$U = \begin{array}{c|cc|cc} & 00 & 01 & 10 & 11 \\ \hline 00 & \multicolumn{2}{c|}{a\begin{pmatrix} e & g \\ f & h \end{pmatrix}} & \multicolumn{2}{c}{c\begin{pmatrix} e & g \\ f & h \end{pmatrix}} \\ 01 & & & & \\ \hline 10 & \multicolumn{2}{c|}{b\begin{pmatrix} e & g \\ f & h \end{pmatrix}} & \multicolumn{2}{c}{d\begin{pmatrix} e & g \\ f & h \end{pmatrix}} \\ 11 & & & & \end{array}$$

$$U_1|0\rangle = \underline{\underline{a|0\rangle}} + b|1\rangle \qquad U_2|0\rangle = e|0\rangle + f|1\rangle.$$

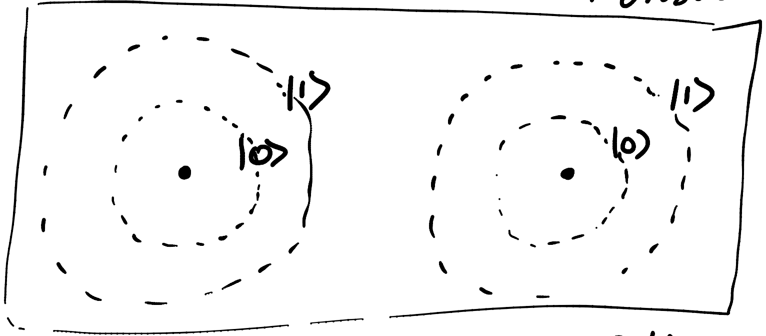
$$(a|0\rangle + b|1\rangle)(e|0\rangle + f|1\rangle) = ae|00\rangle + af|01\rangle$$

$$U_2|1\rangle = g|0\rangle + h|1\rangle$$

$$(a|0\rangle + b|1\rangle)(g|0\rangle + h|1\rangle)$$

$$|01\rangle \longrightarrow$$

Tensor Products.



$|u_1\rangle \otimes |v_1\rangle \qquad |u_2\rangle \otimes |v_2\rangle$

inner product

$$=$$

$$\langle u_1 | u_2 \rangle \cdot \langle v_1 | v_2 \rangle$$

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle \in \mathbb{C}^2 = \mathcal{H}_1 \qquad \beta_0 |0\rangle + \beta_1 |1\rangle \in \mathbb{C}^2 = \mathcal{H}_2$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}$$

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \in \mathbb{C}^4 = \mathcal{H}$$

$|u\rangle \qquad |v\rangle \qquad |u\rangle \otimes |v\rangle$

$|u_1\rangle + |u_2\rangle \qquad |v\rangle \qquad (|u_1\rangle + |u_2\rangle) \otimes |v\rangle = |u_1\rangle \otimes |v\rangle + |u_2\rangle \otimes |v\rangle$

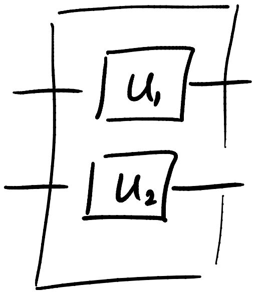
$|u\rangle \qquad |v_1\rangle + |v_2\rangle \longrightarrow$ " "

$|0\rangle \qquad |0\rangle \qquad |0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$

$|0\rangle \qquad |1\rangle \qquad |0\rangle \otimes |1\rangle = |0\rangle|1\rangle = |01\rangle$

$|1\rangle \qquad |0\rangle \qquad |10\rangle$

$|1\rangle \qquad |1\rangle \qquad |11\rangle$



U

$$U_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$U_2 = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$U = U_1 \otimes U_2$$

$$\rotatebox{90}{=}$$

$$\begin{pmatrix} a\,U_2 & c\,U_2 \\ b\,U_2 & d\,l \end{pmatrix}$$