Foundations of Computer Graphics

Online Lecture 2: Review of Basic Math

Vectors and Dot Products

Ravi Ramamoorthi

Course: Next Steps

- Complete HW 0
- Sets up basic compilation issues
- Verifies you can work with feedback/grading servers
- First few lectures core math ideas in graphics
- This lecture is a revision of basic math concepts
- HW 1 has few lines of code (but start early)
- Use some ideas discussed in lecture, create images
- Textbooks: None required
- OpenGL/GLSL reference helpful (but not required)

Motivation and Outline

- Many graphics concepts need basic math like linear algebra
- Vectors (dot products, cross products, …)
- Matrices (matrix-matrix, matrix-vector mult., …)
- E.g: a point is a vector, and an operation like translating or rotating points on object can be matrix-vector multiply
- Should be refresher on very basic material for most of you
- Only basic high school math required

Vectors

- Usually written as \( \mathbf{a} \) or in bold. Magnitude written as \( |\mathbf{a}| \)
- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations
  - But strictly speaking, positions are not vectors and cannot be added: a location implicitly involves an origin, while an offset does not.

Vector Addition

- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

Cartesian Coordinates

\[
\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} x & y \end{pmatrix} \quad \|\mathbf{A}\| = \sqrt{x^2 + y^2}
\]

- \( x \) and \( y \) can be any (usually orthogonal unit) vectors
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
- Note: We use right-handed (standard) coordinates

Dot (scalar) product

\[ a \cdot b = b \cdot a = ? \]

\[ a \cdot b = ||a|| ||b|| \cos \phi \]

\[ \phi = \cos^{-1} \left( \frac{a \cdot b}{||a|| ||b||} \right) \]

Dot product in Cartesian components

\[ a \cdot b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ? \]

\[ a \cdot b = x_a x_b + y_a y_b \]
Dot product: some applications in CG

- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)
- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)
- Advantage: computed easily in cartesian components

Projections (of b on a)

\[ \begin{align*}
\|b \rightarrow a\| &= \frac{a \cdot b}{\|a\|} \\
\|b \rightarrow a\| &= \|b\| \cos \phi \\
b \rightarrow a &= \frac{a \cdot b}{\|a\|^2} a
\end{align*} \]

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Vectors: Cross Products

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Cross (vector) product

- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

Cross product: Properties

\[ \begin{align*}
\mathbf{x} \times \mathbf{y} &= +\mathbf{z} \\
\mathbf{y} \times \mathbf{x} &= -\mathbf{z} \\
\mathbf{y} \times \mathbf{z} &= +\mathbf{x} \\
\mathbf{z} \times \mathbf{y} &= -\mathbf{x} \\
\mathbf{z} \times \mathbf{x} &= +\mathbf{y} \\
\mathbf{x} \times \mathbf{z} &= -\mathbf{y}
\end{align*} \]

Cross product: Cartesian formula?
Cross product: Cartesian formula?

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    a_x & a_y & a_z \\
    b_x & b_y & b_z
\end{vmatrix} = 
\begin{pmatrix}
    a_y b_z - a_z b_y \\
    a_z b_x - a_x b_z \\
    a_x b_y - a_y b_x
\end{pmatrix} = \mathbf{a} \times \mathbf{b} = \mathbf{A}^T \mathbf{b} = \begin{pmatrix}
    0 & -z_a & y_a \\
    z_a & 0 & -x_a \\
    -y_a & x_a & 0
\end{pmatrix}
\]

Dual matrix of vector \( \mathbf{a} \)

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Vectors: Orthonormal Basis Frames
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Orthonormal bases/coordinate frames
- Important for representing points, positions, locations
- Often, many sets of coordinate systems (not just X, Y, Z)
  - Global, local, world, model, parts of model (head, hands, …)
- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Coordinate Frames
- Any set of 3 vectors (in 3D) so that
  \[ ||\mathbf{u}|| = ||\mathbf{v}|| = ||\mathbf{w}|| = 1 \]
  \[ \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0 \]
  \[ \mathbf{w} = \mathbf{u} \times \mathbf{v} \]
  \[ \mathbf{p} = (\mathbf{p} \cdot \mathbf{u})\mathbf{u} + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + (\mathbf{p} \cdot \mathbf{w})\mathbf{w} \]

Constructing a coordinate frame?
- Often, given a vector \( \mathbf{a} \) (viewing direction in HW1), want to construct an orthonormal basis
- Need a second vector \( \mathbf{b} \) (up direction of camera in HW1)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into in HW1)

Constructing a coordinate frame?
- We want to associate \( \mathbf{w} \) with \( \mathbf{a} \), and \( \mathbf{v} \) with \( \mathbf{b} \)
  - But \( \mathbf{a} \) and \( \mathbf{b} \) are neither orthogonal nor unit norm
  - And we also need to find \( \mathbf{u} \)
We want to associate $w$ with $a$, and $v$ with $b$

- But $a$ and $b$ are neither orthogonal nor unit norm
- And we also need to find $u$

$$w = \frac{a}{|a|}$$

$$u = \frac{b \times w}{|b \times w|}$$

- Constructing a coordinate frame?

### Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale
  (more detail next lecture)

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**Matrices**

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### What is a matrix

- Array of numbers ($m \times n = m$ rows, $n$ columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition, multiplication by a scalar simple: element by element
Matrix-matrix multiplication

- Number of columns in first must = rows in second

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
\]

- Element \((i,j)\) in product is dot product of row \(i\) of first matrix and column \(j\) of second matrix

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
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2 & 7 & 8 & 3
\end{pmatrix}
= 
\begin{pmatrix}
9 & 27 & 33 & 13 \\
19 & 44 & 61 & 26 \\
8 & 28 & 32 & 12
\end{pmatrix}
\]

Matrix-matrix multiplication

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Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix \((m \times 1)\)

- E.g. 2D reflection about y-axis

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
-x \\
y
\end{pmatrix}
\]
Transpose of a Matrix (or vector?)

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}^\top = \begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}
\]

\[(AB)^\top = B^\top A^\top\]

Identity Matrix and Inverses

\[
I_{3 \times 3} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[AA^{-1} = A^{-1}A = I\]

\[(AB)^{-1} = B^{-1}A^{-1}\]

Vector multiplication in Matrix form

- Dot product
  \[a \cdot b = a^\top b\]
  \[
  \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
  \cdot
  \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
  = (x_0x_0 + y_0y_0 + z_0z_0)
  \]

- Cross product
  \[a \times b = A b = \begin{pmatrix}
0 & -z_0 & y_0 \\
z_0 & 0 & -x_0 \\
-y_0 & x_0 & 0
\end{pmatrix}\]
  Dual matrix of vector a