Pricing Options with Mathematical Models

1. OVERVIEW

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.
• What we want to accomplish:

Learn the basics of option pricing so you can:
  - (i) continue learning on your own, or in more advanced courses;
  - (ii) prepare for graduate studies on this topic, or for work in industry, or your own business.
• The prerequisites we need to know:

- (i) Calculus based probability and statistics, for example computing probabilities and expected values related to normal distribution.

- (ii) Basic knowledge of differential equations, for example solving a linear ordinary differential equation.

- (iii) Basic programming or intermediate knowledge of Excel
• A rough outline:

- Basic securities: stocks, bonds
- Derivative securities, options
- Deterministic world: pricing fixed cash flows, spot interest rates, forward rates
• A rough outline (continued):

- Stochastic world, pricing options:

  • Pricing by no-arbitrage
  • Binomial trees
  • Stochastic Calculus, Ito’s rule, Brownian motion
  • Black-Scholes formula and variations
  • Hedging
  • Fixed income derivatives
Pricing Options with Mathematical Models

2. Stocks, Bonds, Forwards

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.
A Classification of Financial Instruments

SECURITIES AND CONTRACTS

BASIC SECURITIES
- FIXED INCOME
  - Bonds
- EQUITIES
  - Stocks
- OPTIONS
  - Calls and Puts
- DERIVATIVES AND CONTRACTS
  - SWAPS
  - FUTURES AND FORWARDS
  - CREDIT RISK
  - DERIVATIVES

Loans

Calls and Puts

Exotic Options
Stocks

• Issued by firms to finance operations
• Represent ownership of the firm
• Price known today, but not in the future
• May or may not pay dividends
Bonds

• Price known today
• Future payoffs known at fixed dates
• Otherwise, the price movement is random
• Final payoff at maturity: face value/nominal value/principal
• Intermediate payoffs: coupons
• Exposed to default/credit risk
Derivatives

- Sell for a **price/value/premium** today.
- Future value **derived** from the value of the underlying securities (as a function of those).
- Traded at exchanges – standardized contracts, no credit risk;
- or, over-the-counter (OTC) – a network of dealers and institutions, can be non-standard, some credit risk.
Why derivatives?

- To hedge risk
- To speculate
- To attain “arbitrage” profit
- To exchange one type of payoff for another
- To circumvent regulations
Forward Contract

• An agreement to buy (long) or sell (short) a given underlying asset $S$:
  – At a predetermined future date $T$ (maturity).
  – At a predetermined price $F$ (forward price).

• $F$ is chosen so that the contract has zero value today.

• Delivery takes place at maturity $T$:
  – Payoff at maturity: $S(T) - F$ or $F - S(T)$
  – Price $F$ set when the contract is established.
  – $S(T) =$ spot (market) price at maturity.
Forward Contract (continued)

- Long position: obligation to buy
- Short position: obligation to sell
- Differences with options:
  - Delivery has to take place.
  - Zero value today.
Example

• On May 13, a firm enters into a long forward contract to buy one million euros in six months at an exchange rate of 1.3

• On November 13, the firm pays $1,300,000 and receives S(T) = one million euros.

• How does the payoff look like at time T as a function of the dollar value of S(T) spot exchange rate?
Profit from a long forward position

Profit = \( S(T) - F \)

Value \( S(T) \) of underlying at maturity
Profit from a short forward position

\[ \text{Profit} = F - S(T) \]

Value \( S(T) \) of underlying at maturity
Pricing Options with Mathematical Models

3. Swaps

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.
Swaps

- Agreement between two parties to exchange two series of payments.
- Classic interest rate swap:
  - One party pays **fixed** interest rate payments on a notional amount.
  - Counterparty pays **floating** (random) interest rate payments on the same notional amount.
- Floating rate is often linked to LIBOR (London Interbank Offer Rate), reset at every payment date.
Motivation

• The two parties may be exposed to different interest rates in different markets, or to different institutional restrictions, or to different regulations.
A Swap Example

• New pension regulations require higher investment in fixed income securities by pension funds, creating a problem: liabilities are long-term while new holdings of fixed income securities may be short-term.
• Instead of selling assets such as stocks, a pension fund can enter a swap, exchanging returns from stocks for fixed income returns.
• Or, if it wants to have an option not to exchange, it can buy swaptions instead.
Swap Comparative Advantage

US firm B wants to borrow AUD, Australian firm A wants to borrow USD

Firm B can borrow at 5% in USD, 12.6% AUD
Firm A can borrow at 7% USD, 13% AUD
Expected gain = (7-5) − (13-12.6) = 1.6%

Swap:

← USD5%                    ←USD6.3%
← Firm B                           BANK
  5%    →AUD11.9%               →AUD13% 13%
Firm A→

Bank gains 1.3% on USD, loses 1.1% on AUD, gain=0.2%
Firm B gains (12.6-11.9) = 0.7%
Firm A gains (7-6.3) = 0.7%

Part of the reason for the gain is credit risk involved
A Swap Example: Diversifying

- Charitable foundation CF receives 50mil in stock X from a privately owned firm.
- CF does not want to sell the stock, to keep the firm owners happy
- Equity swap: pays returns on 50mil in stock X, receives return on 50mil worth of S&P500 index.
- A bad scenario: S&P goes down, X goes up; a potential cash flow problem.
Swap Example: Diversifying II

- An executive receives 500mil of stock of her company as compensation.
- She is not allowed to sell.
- Swap (if allowed): pays returns on a certain amount of the stock, receives returns on a certain amounts of a stock index.
- Potential problems: less favorable tax treatment; shareholders might not like it.
Pricing Options with Mathematical Models

4. Call and Put Options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.
Vanilla Options

– **Call** option: a right to buy the underlying

– **Put** option: a right to sell the underlying

– **European** option: the right can be *exercised* only at *maturity*

– **American** option: can be exercised at any time before maturity
Various underlying variables

- Stock options
- Index options
- Futures options
- Foreign currency options
- Interest rate options
- Credit risk derivatives
- Energy derivatives
- Mortgage based securities
- Natural events derivatives …
Exotic options

– **Asian options**: the payoff depends on the average underlying asset price

– **Lookback options**: the payoff depends on the maximum or minimum of the underlying asset price

– **Barrier options**: the payoff depends on whether the underlying crossed a barrier or not

– **Basket options**: the payoff depends on the value of several underlying assets.
Terminology

- **Writing an option**: selling the option
- **Premium**: price or value of an option
- Option **in/at/out of the money**:
  - *At*: strike price equal to underlying price
  - *In*: immediate exercise would be profitable
  - *Out*: immediate exercise would not be profitable
## Long Call

Outcome at maturity

<table>
<thead>
<tr>
<th></th>
<th>(S(T) \leq K)</th>
<th>(S(T) &gt; K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>(0)</td>
<td>(S(T) - K)</td>
</tr>
<tr>
<td>Profit</td>
<td>(-C(t, K, T))</td>
<td>(S(T) - K - C(t, K, T))</td>
</tr>
</tbody>
</table>

A more compact notation:

Payoff: \(\max [S(T) - K, 0] = (S(T)-K)^+\)

Profit: \(\max [S(T) - K, 0] - C(t, K, T)\)
Long Call Position

- Assume $K = $50, $C(t, K, T) = $6
- Payoff: $\max [S(T) - 50, 0]$
- Profit: $\max [S(T) - 50, 0] - 6$

Payoff Profit

$S(T) = K = 50$

Break-even:

$S(T) = 56$

$-6$

$S(T)$
Short Call Position

- $K = \$50$, $C(t,K,T) = \$6$
- Payoff: $-\max [S(T) - 50, 0]$
- Profit: $6 - \max [S(T) - 50, 0]$

Payoff: \(S(T)=K=50\)

Profit: \(S(T)=K=50\)

Break-even: \(S(T)=56\)
## Long Put

### Outcome at maturity

<table>
<thead>
<tr>
<th>Condition</th>
<th>Payoff</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K$</td>
<td>$K - S(T)$</td>
<td>$K - S(T) - P(t, K, T)$</td>
</tr>
<tr>
<td>$S(T) &gt; K$</td>
<td>0</td>
<td>$- P(t, K, T)$</td>
</tr>
</tbody>
</table>

### A more compact notation:

**Payoff:**\[\max \left[ K - S(T), 0 \right] = (K - S(T))^+\]

**Profit:**\[\max \left[ K - S(T), 0 \right] - P(t, K, T)\]
Long Put Position

- Assume $K = $50, $P(t,K,T) = $8
- Payoff: $\max [50 - S(T), 0]$
- Profit: $\max [50 - S(T), 0] - 8$

Payoff

Profit

Break-even: $S(T)=42$

$S(T)=K=50$

$S(T)=K=50$

$S(T)$

$S(T)$
**Short Put Position**

- $K = $50, $P(t,K,T) = $8$
- Payoff: $- \max [50 - S(T), 0]$
- Profit: $8 - \max [50 - S(T), 0]$

Payoff

Profit

Break-even: $S(T) = 42$

$S(T) = K = 50$

$S(T) = K = 50$

$S(T) = K = 50$

$S(T) = K = 50$
Implicit Leverage: Example

• Consider two securities
  – Stock with price $S(0) = $100$
  – Call option with price $C(0) = $2.5 (K = $100)$

• Consider three possible outcomes at $t=T$:
  – Good: $S(T) = $105$
  – Intermediate: $S(T) = $101$
  – Bad: $S(T) = $98$
Suppose we plan to invest $100

<table>
<thead>
<tr>
<th>Invest in:</th>
<th>Stocks</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Return in:</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Good State</td>
<td>5%</td>
<td>100%</td>
</tr>
<tr>
<td>Mid State</td>
<td>1%</td>
<td>-60%</td>
</tr>
<tr>
<td>Bad State</td>
<td>-2%</td>
<td>-100%</td>
</tr>
</tbody>
</table>
EQUITY LINKED BANK DEPOSIT

- Investment = 10,000
- Return = 10,000 if an index below the current value of 1,300 after 5.5 years
- Return = 10,000 \times (1 + 70\% \text{ of the percentage return on index})
- Example: Index = 1,500. Return = 10,000 \cdot (1 + (1,500/1,300-1) \cdot 70\%) = 11,077
- Payoff = Bond + call option on index
HEDGING EXAMPLE

Your bonus compensation: 100 shares of the company, each worth $150.

Your hedging strategy: buy 50 put options with strike $K = 150$

If share value falls to $100: you lose $5,000 in stock, win $2,500 minus premium in options
Pricing Options with Mathematical Models

5. Options Combinations

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.
Bull Spread Using Calls

\[ S(T) \]

Profit

\[ K_1 \quad K_2 \]
Bull Spread Using Puts

Profit

$K_1 \quad K_2 \quad S(T)$
Bear Spread Using Puts

Profit

\[ K_1 \quad K_2 \quad S(T) \]
Bear Spread Using Calls

![Diagram of Bear Spread Using Calls]

- $K_1$
- $K_2$
- $S(T)$
- Profit
Butterfly Spread Using Calls

Profit

$K_1 \quad K_2 \quad K_3 \quad S(T)$
Butterfly Spread Using Puts

Profit

$K_1 \quad K_2 \quad K_3 \quad S(T)$
Bull Spread (Calls)

- Two strike prices: $K_1, K_2$ with $K_1 < K_2$
- Short-hand notation: $C(K_1), C(K_2)$

Outcome at Expiration

<table>
<thead>
<tr>
<th>$S(T)$ ≤ $K_1$</th>
<th>$K_1 &lt; S(T) ≤ K_2$</th>
<th>$S(T) &gt; K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff: 0</td>
<td>$S(T) - K_1$</td>
<td>$S(T) - K_1 - (S(T) - K_2) = K_2 - K_1$</td>
</tr>
<tr>
<td>Profit: $C(K_2) - C(K_1)$</td>
<td>$C(K_2) - C(K_1) + S(T) - K_1$</td>
<td>$C(K_2) - C(K_1) + K_2 - K_1$</td>
</tr>
</tbody>
</table>
Bull Spread (Calls)

- Assume $K_1 = $50, $K_2 = $60, $C(K_1) = $10, $C(K_2) = $6
- Payoff: $\max [S(T) - 50, 0] - \max [S(T) - 60, 0]$
- Profit: $(6-10) + \max [S(T)-50,0] - \max [S(T)-60,0]$

Payoff Profit

$K_1=50$ $K_2=60$

Break-even: $S(T)=54$

$K_2=60$
## Bear Spread (Puts)

- Again two strikes: $K_1, K_2$ with $K_1 < K_2$
- Short-hand notation: $P(K_1), P(K_2)$

### Outcome at Expiration

<table>
<thead>
<tr>
<th>$S(T) \leq K_1$</th>
<th>$K_1 &lt; S(T) \leq K_2$</th>
<th>$S(T) &gt; K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Payoff:</strong></td>
<td><strong>Profit:</strong></td>
<td><strong>Profit:</strong></td>
</tr>
<tr>
<td>$K_2 - S(T) - (K_1 - S(T)) = K_2 - K_1$</td>
<td>$P(K_1) - P(K_2) + K_2 - K_1$</td>
<td>$P(K_1) - P(K_2) + K_2 - S(T)$</td>
</tr>
</tbody>
</table>
Calendar Spread

Payoff

\[ S(T) \]

Short Call \((T_1)\) + Long Call \((T_2)\)

\[ K \]
Butterfly Spread

• Positions in **three** options of the same class, with same maturities but different strikes $K_1$, $K_2$, $K_3$

  – Long butterfly spreads: buy one option each with strikes $K_1$, $K_3$, sell two with strike $K_2$

• $K_2 = (K_1 + K_3) / 2$
Long Butterfly Spread (Puts)

- $K_1 = $50, $K_2 = $55, $K_3 = $60
- $P(K_1) = $4, $P(K_2) = $6, $P(K_3) = $10

Payoff Profit

- Break-even 1: $S(T) = 52$
- Break-even 2: $S(T) = 58$
Bottom Straddle

Assume $K = $50, $P(K) = $8, $C(K) = $6

Payoff Profit

$K = 50$

$S(T)$

Profit

$K = 50$

$S(T)$

Break-even 1: $S(T) = 36$

Break-even 2: $S(T) = 64$

$-14$
Bottom Strangle

Assume $K_1 = $50, $K_2 = $60, $P(K_1) = $8, $C(K_2) = $6

Payoff Profit

\[ K = 50 \]
\[ S(T) \]
\[ 50 \]

Break-even 1: $S(T) = 36$

Break-even 2: $S(T) = 74$

Profit

\[ K = 50 \]
\[ K = 60 \]
\[ -14 \]

\[ 36 \]
Arbitrary payoff shape

- Suppose we want to have a payoff of the form $f(S(T))$ for some function $f()$. Assume that call options written on $S(T)$ are traded for all possible strike values $K$.
- CLAIM: If $f()$ is smooth and $f'(\infty) \cdot 0 = 0$, then

\[
f(s) = f(0) + f'(0)s + \int_0^\infty f''(K) \max(S - K, 0) \, dK
\]
Proof sketch

\[ \int_0^\infty f''(K) \max(s - K, 0) \, dK \]

= (integration by parts) =

= \left[ f'(\infty) \cdot 0 - f'(0) \cdot s - \int_0^\infty f'(K) \, d[\max(s - K, 0)] \right]

= - f'(0) \cdot s + \int_0^s f'(K) \, dK

= - f'(0) \cdot s + f(s) - f(0) .